

The structure of $\mathcal{N} = 2$ supersymmetric nonlinear sigma models in AdS_4

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Abstract

We present a detailed study of the most general $\mathcal{N} = 2$ supersymmetric sigma models in four-dimensional anti-de Sitter space (AdS_4) formulated in terms of $\mathcal{N} = 1$ chiral superfields. The target space is demonstrated to be a non-compact hyperkähler manifold restricted to possess a special Killing vector field which generates an $\text{SO}(2)$ group of rotations on the two-sphere of complex structures and necessarily leaves one of them invariant. All hyperkähler cones, that is the target spaces of $\mathcal{N} = 2$ superconformal sigma models, prove to possess such a vector field that belongs to the Lie algebra of an isometry group $\text{SU}(2)$ acting by rotations on the complex structures. A unique property of the $\mathcal{N} = 2$ sigma models constructed is that the algebra of $\text{OSp}(2|4)$ transformations closes off the mass shell. We uncover the underlying $\mathcal{N} = 2$ superfield formulation for the $\mathcal{N} = 2$ sigma models constructed and compute the associated $\mathcal{N} = 2$ supercurrent.

We give a special analysis of the most general systems of self-interacting $\mathcal{N} = 2$ tensor multiplets in AdS_4 and their dual sigma models realized in terms of $\mathcal{N} = 1$ chiral multiplets. We also briefly discuss the relationship between our results on $\mathcal{N} = 2$ supersymmetric sigma models formulated in the $\mathcal{N} = 1$ AdS superspace and the off-shell sigma models constructed in the $\mathcal{N} = 2$ AdS superspace in arXiv:0807.3368.

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1 Introduction

In four space-time dimensions, Poincaré supersymmetry is intimately connected to complex geometry. The target space \mathcal{M} of a rigid supersymmetric σ -model is a Kähler manifold in the case of $\mathcal{N} = 1$ supersymmetry [1] and a hyperkähler space for $\mathcal{N} = 2$ [2] (see also [3]). Extending the Poincaré supersymmetry to superconformal symmetry proves to add the requirement that \mathcal{M} be endowed with a homothetic conformal Killing vector which is the gradient of a function [4] (this is most transparent in three dimensions [5, 6]), and thus \mathcal{M} is globally a cone [7]. The target spaces of rigid superconformal σ -models are Kähler cones in the case $\mathcal{N} = 1$, and hyperkähler cones for $\mathcal{N} = 2$ [8, 9, 10].

There exist two standard approaches to describe the most general $\mathcal{N} = 1$ supersymmetric σ -models: (i) in terms of component physical fields; or (ii) in terms of $\mathcal{N} = 1$ chiral superfields. The latter approach is more efficient, due to its intrinsically geometric form and off-shell supersymmetry.

In the case of $\mathcal{N} = 2$ rigid supersymmetric σ -models, it is also natural to make use of a formulation that permits some amount of supersymmetry to be realized manifestly. This requires the use of either $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superspace techniques. In the past, two powerful $\mathcal{N} = 2$ superspace approaches were developed to construct off-shell σ -models, which are: (i) the harmonic superspace [11, 12]; and (ii) the projective superspace [13, 14]. One of their conceptual virtues is the possibility to generate supersymmetric σ -model actions (and thus hyperkähler metrics) from Lagrangians of *arbitrary* functional form. Still, many supersymmetry practitioners consider a formulation in $\mathcal{N} = 1$ superspace as the most transparent and economical one.

In 1986, Hull et al. [15] formulated, building on the earlier work of Lindström and Roček [16], the most general $\mathcal{N} = 2$ rigid supersymmetric σ -models without superpotentials in terms of $\mathcal{N} = 1$ chiral superfields. An extension of [15] to include superpotentials was given in [17]. Arbitrary $\mathcal{N} = 2$ superconformal σ -models were described in terms of $\mathcal{N} = 1$ chiral superfields in [18].

Recently, there has been a renewed interest in supersymmetric field theories in four-dimensional anti-de Sitter space (AdS_4) [19, 20, 21]. This motivated us in [22] to construct, as a nontrivial extension of [15], the most general $\mathcal{N} = 2$ AdS supersymmetric σ -models in terms of covariantly chiral superfields on $\mathcal{N} = 1$ AdS superspace.¹ In the present work we elaborate upon the results announced in [22] by providing

¹Historically, the $\mathcal{N} = 1$ AdS superspace, $\text{AdS}^{4|4} := \text{OSp}(1|4)/\text{O}(3, 1)$, was introduced in [23, 24], and the superfield approach to $\text{OSp}(1|4)$ supersymmetry was developed by Ivanov and Sorin [25].

technical proofs and detailed explanations. Moreover, we considerably extend [22] by including new results. In particular, we give a special analysis of the most general systems of self-interacting $\mathcal{N} = 2$ tensor multiplets in AdS_4 and their dual σ -models realized in terms of $\mathcal{N} = 1$ chiral multiplets. We also develop a manifestly $\mathcal{N} = 2$ superfield formulation corresponding to the nonlinear σ -model constructed in [22].

It should be mentioned that off-shell supersymmetric σ -models in the $\mathcal{N} = 2$ AdS superspace have already been constructed in [26], building on the projective superspace formulation for general $\mathcal{N} = 2$ supergravity-matter systems [27, 28]. For the series of $\mathcal{N} = 2$ σ -models presented in [26], one can in principle derive their reformulation in terms of $\mathcal{N} = 1$ chiral superfields by (i) eliminating the (infinitely many) auxiliary superfields; and (ii) performing appropriate superspace duality transformations. These are nontrivial technical problems. Their solution should be similar in spirit to the analysis given in [18], but explicit calculations remain to be done in the future. Here we only briefly discuss the relationship between our results on $\mathcal{N} = 2$ supersymmetric σ -models formulated in the $\mathcal{N} = 1$ AdS superspace and the off-shell σ -models constructed in the $\mathcal{N} = 2$ AdS superspace [26].

This paper is organized as follows. In the first half, we present an analysis of the properties of $\mathcal{N} = 2$ tensor multiplet actions when written in terms of $\mathcal{N} = 1$ superfields. Section 2 provides a brief review of the topic in Minkowski space. Section 3 deals with superconformal tensor multiplets, emphasizing their properties in $\mathcal{N} = 1$ superspace. We address the AdS situation in section 4 and derive an additional condition on the Lagrangian necessary for $\mathcal{N} = 2$ supersymmetry in AdS. Finally in section 5 we briefly discuss how the new constraint required in AdS is naturally understood from a projective superspace setting.

The second half of the paper is devoted to general σ -models involving hypermultiplets, which are represented purely in terms of $\mathcal{N} = 1$ chiral multiplets. In section 6, we analyze the conditions for $\mathcal{N} = 2$ supersymmetry in AdS and show that the target space must be hyperkähler and possess a special Killing vector with quite interesting properties. We further analyze the geometric implications in section 7 and briefly discuss how the AdS condition is naturally fulfilled by superconformal σ -models in section 8. In section 9 we discuss a novel superfield formulation of the hypermultiplet in AdS and in section 10 we briefly discuss the form of the supercurrent.

There are four technical appendices. The first deals with $\mathcal{N} = 1$ superconformal Killing vector fields in an AdS background; the second presents the analysis for $\mathcal{N} = 2$. The third appendix provides an alternative technical proof of $\mathcal{N} = 2$ supersymmetry for nonlinear σ -models which is more direct than the one offered in subsection 6.4.

The last addresses a technical issue of the non-minimal supercurrent in AdS.

2 Self-interacting $\mathcal{N} = 2$ tensor supermultiplets: Poincaré supersymmetry

The $\mathcal{N} = 2$ tensor multiplet [29, 30, 31, 32, 33] consists of an $SU(2)$ triplet of scalars g_{ij} , a gauge two-form b_{mn} , a complex scalar F , and a doublet of Weyl fermions $\chi_{\alpha i}$. This set of component fields gives an off-shell representation of $\mathcal{N} = 2$ supersymmetry. In $\mathcal{N} = 2$ superspace it is described [31, 33] by an iso-triplet superfield \mathcal{G}^{ij} which has the algebraic properties $\bar{\mathcal{G}}_{ij} := (\mathcal{G}^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}\mathcal{G}^{kl}$, and obeys the constraints

$$D_{\alpha}^{(i}\mathcal{G}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i}\mathcal{G}^{jk)} = 0 . \quad (2.1)$$

Upon reduction to $\mathcal{N} = 1$ superspace, \mathcal{G}^{ij} decomposes into a real linear superfield G and a chiral scalar φ .² This is why it is also called the $\mathcal{N} = 2$ linear multiplet [31].

Models of several $\mathcal{N} = 2$ tensor multiplets can therefore be realized in $\mathcal{N} = 1$ superspace, where the $\mathcal{N} = 1$ content involves a set of chiral superfields φ^I , their conjugates $\bar{\varphi}^I$ and a set of real linear superfields $G^I = \bar{G}^I$, obeying the usual constraints

$$\bar{D}_{\dot{\alpha}}\varphi^I = 0 , \quad \bar{D}^2 G^I = D^2 G^I = 0 . \quad (2.2)$$

Here the index I runs over the full set of n tensor multiplets, $I = 1, \dots, n$.

Our goal in this *review* section is to construct the most general nonlinear σ -models³

$$S = \int d^4x d^4\theta L(\varphi^I, \bar{\varphi}^I, G^I) \quad (2.3)$$

which are invariant under the second supersymmetry transformations [16]

$$\delta\varphi^I = \bar{\epsilon}\bar{D}G^I , \quad (2.4a)$$

$$\delta G^I = -\epsilon D\varphi^I - \bar{\epsilon}\bar{D}\bar{\varphi}^I , \quad (2.4b)$$

$$\delta\bar{\varphi}^I = \epsilon DG^I . \quad (2.4c)$$

This problem was solved by Lindström and Roček in 1983 [16], but the technical details of the derivation were not included. Here we give a complete derivation in

²The real linear superfield G is used to describe the $\mathcal{N} = 1$ tensor multiplet [34].

³Recall that the Grassmann measure is defined $d^4\theta := d^2\theta d^2\bar{\theta}$.

superspace which is interesting in its own right, and more importantly can be generalized to the superconformal and AdS cases. Along the way we will rediscover certain interesting features of such σ -models which were noticed in [10] for the case of superconformal tensor models.

One important feature of the Lagrangian (2.3) to keep in mind is that it possesses two classes of trivial transformations

$$L \rightarrow L + F(\varphi) + \bar{F}(\bar{\varphi}) , \quad (2.5a)$$

$$L \rightarrow L + G^I H_I(\varphi) + G^I \bar{H}_I(\bar{\varphi}) \quad (2.5b)$$

under which the action is invariant. The first is a Kähler-like transformation and is particular to the Minkowski case alone. The second, which we denote the H -transformation, has a much broader applicability, generalizing not only to AdS but also to arbitrary $\mathcal{N} = 1$ supergravity backgrounds.

2.1 Conditions for $\mathcal{N} = 2$ supersymmetry

Let us work out what conditions L must obey in order to be $\mathcal{N} = 2$ supersymmetric. Since the supersymmetry parameters ϵ^α and $\bar{\epsilon}_{\dot{\alpha}}$ in (2.4) are independent, it is sufficient to analyze the condition of invariance under the $\bar{\epsilon}$ -transformation which is obtained from (2.4) by setting $\epsilon^\alpha = 0$. This condition is

$$\delta_{\bar{\epsilon}} S = \int d^4x d^4\theta \left\{ \frac{\partial L}{\partial \varphi^I} \bar{\epsilon} \bar{D} G^I - \frac{\partial L}{\partial G^I} \bar{\epsilon} \bar{D} \bar{\varphi}^I \right\} = 0 , \quad (2.6)$$

with $\bar{\epsilon}_{\dot{\alpha}}$ a constant anti-commuting parameter.

The requirement (2.6) means that the functional must vanish identically for arbitrary values of the superfields. Let us vary (2.6) with respect to $\bar{\varphi}^J$. This gives

$$\int d^4x d^4\theta \delta \bar{\varphi}^J \left\{ \left(\frac{\partial^2 L}{\partial \bar{\varphi}^J \partial \varphi^I} + \frac{\partial^2 L}{\partial G^J \partial G^I} \right) \bar{\epsilon} \bar{D} G^I + \left(\frac{\partial^2 L}{\partial G^J \partial \bar{\varphi}^I} - \frac{\partial^2 L}{\partial G^I \partial \bar{\varphi}^J} \right) \bar{\epsilon} \bar{D} \bar{\varphi}^I \right\} . \quad (2.7)$$

For this to vanish, we are forced to require that L obey the Laplace equation [16]

$$\frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial G^I \partial G^J} = 0 . \quad (2.8)$$

This equation turns out to imply that the remaining expression in (2.7),

$$\int d^4x d^4\theta \left(\frac{\partial^2 L}{\partial G^J \partial \bar{\varphi}^I} - \frac{\partial^2 L}{\partial G^I \partial \bar{\varphi}^J} \right) \delta \bar{\varphi}^J \bar{\epsilon} \bar{D} \bar{\varphi}^I , \quad (2.9)$$

is indeed equal to zero. To prove this assertion, we have to make two observations. Firstly, the equation (2.8) implies that

$$\frac{\partial}{\partial G^K} \left(\frac{\partial^2 L}{\partial G^J \partial \varphi^I} - \frac{\partial^2 L}{\partial G^I \partial \varphi^J} \right) = \frac{\partial}{\partial \bar{\varphi}^K} \left(\frac{\partial^2 L}{\partial G^J \partial \varphi^I} - \frac{\partial^2 L}{\partial G^I \partial \varphi^J} \right) = 0 , \quad (2.10)$$

and hence

$$\frac{\partial^2 L}{\partial G^J \partial \varphi^I} - \frac{\partial^2 L}{\partial G^I \partial \varphi^J} = F_{IJ}(\varphi) = -F_{JI}(\varphi) , \quad (2.11)$$

for some holomorphic two-form $F_{IJ}(\varphi)$ [10]. Secondly, making use of the integration rules

$$\int d^4x d^4\theta U = -\frac{1}{4} \int d^4x d^2\theta \bar{D}^2 U = -\frac{1}{4} \int d^4x d^2\bar{\theta} D^2 U \quad (2.12)$$

we may show that (2.9) vanishes,

$$\int d^4x d^4\theta \bar{F}_{IJ}(\bar{\varphi}) \delta \bar{\varphi}^J \bar{\epsilon} \bar{D} \bar{\varphi}^I = -\frac{1}{4} \int d^4x d^2\bar{\theta} \bar{F}_{IJ}(\bar{\varphi}) \delta \bar{\varphi}^J \bar{\epsilon}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \bar{\varphi}^I \equiv 0 , \quad (2.13)$$

since $\bar{\varphi}^I$ is antichiral.

It follows from the definition of the two-form $F_{IJ}(\varphi)$, eq. (2.11), that it is closed,

$$\partial_K F_{IJ} + \partial_I F_{JK} + \partial_J F_{KI} = 0 . \quad (2.14)$$

On the other hand, eq. (2.11) tells us that this two-form can be written as

$$F_{IJ} = \partial_I \tilde{H}_J - \partial_J \tilde{H}_I , \quad \tilde{H}_I := \frac{\partial}{\partial G^I} L(\varphi, \bar{\varphi}, G) . \quad (2.15)$$

Since $F_{IJ}(\varphi)$ does not depend on $\bar{\varphi}$ and G , we can choose these variables appearing in $\tilde{H}_I(\varphi, \bar{\varphi}, G)$ to have any given values, say $\bar{\varphi}_0$ and G_0 . Then the above relation turns into

$$F_{IJ}(\varphi) = \partial_I H_J(\varphi) - \partial_J H_I(\varphi) , \quad H_I(\varphi) := \frac{\partial}{\partial G^I} L(\varphi, \bar{\varphi}_0, G_0) . \quad (2.16)$$

Now, we can perform the following transformation of the Lagrangian

$$L \longrightarrow L - G^I H_I(\varphi) - G^I \bar{H}_I(\bar{\varphi}) . \quad (2.17)$$

This transformation does not change the action, since it is of the type (2.5b). Due to (2.11), the transformed Lagrangian obeys the equation

$$\frac{\partial^2 L}{\partial G^I \partial \varphi^J} - \frac{\partial^2 L}{\partial G^J \partial \varphi^I} = 0 . \quad (2.18)$$

As a result, we can always choose the Lagrangian to obey (2.18) [10].

With the aid of the equations (2.8) and (2.18), it is not difficult to prove that the variation $\delta_{\bar{\epsilon}}S$, eq. (2.6), vanishes identically. Making use of the identity (2.12), we represent $\delta_{\bar{\epsilon}}S$ as an integral over the chiral subspace:

$$\delta_{\bar{\epsilon}}S = -\frac{1}{4} \int d^4x d^2\theta \bar{D}^2 \left\{ \frac{\partial L}{\partial \varphi^I} \bar{\epsilon} \bar{D} G^I - \frac{\partial L}{\partial G^I} \bar{\epsilon} \bar{D} \bar{\varphi}^I \right\} . \quad (2.19)$$

Evaluating the integrand and making use of the equations (2.8) and (2.18), we indeed observe that $\delta_{\bar{\epsilon}}S = 0$. The calculation is somewhat longer if one does not require eq. (2.18) to hold, but instead one has to use only the weaker equation (2.11). Such a calculation also gives $\delta_{\bar{\epsilon}}S = 0$.

2.2 Projective superspace formulation

In 1984, the most general self-couplings of several $\mathcal{N} = 2$ tensor multiplets were constructed using $\mathcal{N} = 2$ superspace techniques [13]. Their manifestly $\mathcal{N} = 2$ supersymmetric action can be rewritten in terms of $\mathcal{N} = 1$ superfields, and the result obtained in [13] is

$$S = \text{Re} \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \mathcal{L}(\mathcal{G}^I(\zeta), \zeta) , \quad (2.20)$$

for an appropriately chosen closed contour γ in the complex projective space \mathbb{CP}^1 parametrized by an inhomogeneous complex variable ζ . The dynamical variables in (2.20) are:

$$\mathcal{G}^I(\zeta) = \frac{1}{\zeta} \varphi^I + G^I - \zeta \bar{\varphi}^I , \quad \bar{D}_{\dot{\alpha}} \varphi^I = 0 , \quad \bar{D}^2 G^I = \bar{G}^I - G^I = 0 . \quad (2.21)$$

The Lagrangian in (2.20) is an arbitrary analytic function of its arguments. Evaluating the contour integral, we find the $\mathcal{N} = 1$ action (2.3) with Lagrangian

$$L(\varphi^I, \bar{\varphi}^I, G^I) = \text{Re} \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \mathcal{L}(\mathcal{G}^I(\zeta), \zeta) . \quad (2.22)$$

Using this representation, it is easy to see that both the equations (2.8) and (2.18) hold *automatically*.⁴ In fact, one can also check that this formulation *automatically* selects out Lagrangians with vanishing F_{IJ} (2.11).

⁴In the case of a single $\mathcal{N} = 2$ tensor multiplet, eq. (2.22) coincides with Whittaker's formula for harmonic functions in \mathbb{R}^3 [35].

2.3 Dual formulation

To construct a dual formulation of the theory (2.3), we follow [16] (see also [34]) and associate with (2.3) the first-order action

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ L(\varphi, \bar{\varphi}, V) - V^I(\psi_I + \bar{\psi}_I) \right\}, \quad (2.23)$$

where the real superfields V^I are unconstrained, and the Lagrange multipliers ψ_I are chiral, $\bar{D}_{\dot{\alpha}}\psi_I = 0$. This formulation is equivalent to the original theory, since varying ψ_I and its conjugate $\bar{\psi}_I$ gives the equations $\bar{D}^2 V^I = D^2 V^I = 0$, and then (2.23) turns into (2.3). On the other hand, varying (2.23) with respect to V^I gives the equations

$$\frac{\partial}{\partial V^I} L(\varphi, \bar{\varphi}, V) = \psi_I + \bar{\psi}_I \quad (2.24)$$

which can be used to express V^I in terms of the other variables, $V^I = V^I(\varphi, \bar{\varphi}, \psi + \bar{\psi})$. As a result, we end up with the dual action

$$S_{\text{dual}} = \int d^4x d^4\theta K(\varphi^I, \bar{\varphi}^I, \psi_J + \bar{\psi}_J), \quad (2.25)$$

where the Kähler potential is defined by

$$K(\varphi, \bar{\varphi}, \psi + \bar{\psi}) := \left\{ L(\varphi, \bar{\varphi}, V) - V^I(\psi_I + \bar{\psi}_I) \right\} \Big|_{V=V(\varphi, \bar{\varphi}, \psi + \bar{\psi})}. \quad (2.26)$$

The Kähler potential depends on ψ_I and $\bar{\psi}_I$ only in the combination $\psi_I + \bar{\psi}_I$, and therefore the target space has at least n Abelian isometries.

Assuming that eq. (2.18) holds, the first-order action (2.23) can be shown to be invariant under the second supersymmetry transformations

$$\delta\varphi^I = \frac{1}{2}\bar{D}^2(\bar{\epsilon}\bar{\theta}V^I), \quad (2.27a)$$

$$\delta V^I = -\epsilon D\varphi^I - \bar{\epsilon}\bar{D}\bar{\varphi}^I, \quad (2.27b)$$

$$\delta\psi_I = \frac{1}{2}\bar{D}^2(\bar{\epsilon}\bar{\theta}\frac{\partial L}{\partial\varphi^I}). \quad (2.27c)$$

For the dual model, eq. (2.25), this invariance turns into

$$\delta\varphi^I = -\frac{1}{2}\bar{D}^2(\bar{\epsilon}\bar{\theta}\frac{\partial K}{\partial\psi_I}), \quad (2.28a)$$

$$\delta\psi_I = \frac{1}{2}\bar{D}^2(\bar{\epsilon}\bar{\theta}\frac{\partial K}{\partial\varphi^I}). \quad (2.28b)$$

In the general case, when only the equation (2.11) holds instead of (2.18), the supersymmetry transformation (2.28) takes the form

$$\delta\varphi^I = -\frac{1}{2}\bar{D}^2\left(\bar{\epsilon}\bar{\theta}\frac{\partial K}{\partial\psi_I}\right), \quad (2.29a)$$

$$\delta\psi_I = \frac{1}{2}\bar{D}^2\left(\bar{\epsilon}\bar{\theta}\frac{\partial K}{\partial\varphi^I} + \bar{\epsilon}\bar{\theta}F_{IJ}\frac{\partial K}{\partial\psi_J}\right). \quad (2.29b)$$

Since the σ -model (2.25) is $\mathcal{N} = 2$ supersymmetric and realized in terms of chiral superfields, the Lagrangian in (2.25) is the Kähler potential of a hyperkähler manifold.

3 $\mathcal{N} = 2$ superconformal tensor supermultiplets

It is of interest to extend the above analysis to the case of $\mathcal{N} = 1$ superconformal tensor multiplets. The superconformal couplings of $\mathcal{N} = 2$ tensor multiplets were systematically discussed in the component approach in [10]. Within the $\mathcal{N} = 2$ projective superspace approach [13, 14], they were studied in [13, 36, 37]. We are not aware of a detailed discussion of the general superconformal σ -models of $\mathcal{N} = 2$ tensor multiplets in $\mathcal{N} = 1$ superspace.

It was shown in [36] that the general $\mathcal{N} = 2$ superconformal transformation decomposes, upon reduction to $\mathcal{N} = 1$ superspace, into three types of $\mathcal{N} = 1$ transformations:

1. An arbitrary $\mathcal{N} = 1$ superconformal transformation generated by

$$\xi = \bar{\xi} = \xi^a(z)\partial_a + \xi^\alpha(z)D_\alpha + \bar{\xi}_{\dot{\alpha}}(z)\bar{D}^{\dot{\alpha}} \quad (3.1)$$

such that

$$[\xi, D_\alpha] = -\lambda_\alpha{}^\beta D_\beta + \left(\frac{1}{2}\sigma - \bar{\sigma}\right)D_\alpha \implies \bar{D}_{\dot{\alpha}}\sigma = 0, \quad \bar{D}_{\dot{\alpha}}\lambda_\alpha{}^\beta = 0. \quad (3.2)$$

The lowest component of $\text{Re } \sigma$ corresponds to a dilatation, $\text{Im } \sigma$ to a $U(1)_R$ rotation, and $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ to a Lorentz transformation. (Further details may be found in [38].) This transformation acts on the $\mathcal{N} = 1$ components of the $\mathcal{N} = 2$ tensor multiplet as

$$\delta\varphi^I = -\xi\varphi^I - 2\sigma\varphi^I, \quad (3.3a)$$

$$\delta G^I = -\xi G^I - (\sigma + \bar{\sigma})G^I. \quad (3.3b)$$

2. An extended superconformal transformation generated by a spinor parameter ρ^α constrained as

$$\bar{D}_{\dot{\alpha}}\rho^\beta = 0 \ , \quad D^{(\alpha}\rho^{\beta)} = 0 \ , \quad (3.4)$$

and hence

$$\partial^{\dot{\alpha}(\alpha}\rho^{\beta)} = D^2\rho^\beta = 0 \ . \quad (3.5)$$

The general solution to (3.4) is

$$\rho^\alpha(x_{(+)}, \theta) = \epsilon^\alpha + \lambda\theta^\alpha - i\bar{\eta}_{\dot{\alpha}}x_{(+)}^{\dot{\alpha}\alpha} \ , \quad x_{(+)}^a = x^a + i\theta\sigma^a\bar{\theta} \ . \quad (3.6)$$

Here the *constant* parameters ϵ^α , λ and $\bar{\eta}_{\dot{\alpha}}$ correspond to (i) a second Q-supersymmetry transformation (ϵ^α); (ii) an off-diagonal SU(2)-transformation⁵ (λ); and (iii) a second S-supersymmetry transformation ($\bar{\eta}_{\dot{\alpha}}$). The extended superconformal transformation acts on the $\mathcal{N} = 1$ components, φ^I and G^I , of the $\mathcal{N} = 2$ tensor multiplet as

$$\delta\varphi^I = \bar{\rho}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}G^I + \frac{1}{2}(\bar{D}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}})G^I \ , \quad (3.7a)$$

$$\delta G^I = -D^\alpha(\rho_\alpha\varphi^I) - \bar{D}_{\dot{\alpha}}(\bar{\rho}^{\dot{\alpha}}\bar{\varphi}^I) \ . \quad (3.7b)$$

3. A shadow chiral rotation generated by a constant parameter α . In $\mathcal{N} = 2$ superspace, this is a phase transformation of $\theta_{\underline{2}}^\alpha$ only, with $\theta_{\underline{1}}^\alpha$ kept unchanged. Its action on the $\mathcal{N} = 2$ tensor multiplet is

$$\delta\varphi^I = i\alpha\varphi^I \ , \quad \delta G^I = 0 \ . \quad (3.8)$$

We have seen that under the conditions (2.8) and (2.18), the action (2.3) is $\mathcal{N} = 2$ supersymmetric. Now we would like to determine conditions for $\mathcal{N} = 2$ superconformal invariance. The action proves to be invariant under the $\mathcal{N} = 1$ superconformal transformations (3.3) and the shadow chiral rotations (3.8) if the Lagrangian obeys the following equations:

$$\left(G^I\frac{\partial}{\partial G^I} + 2\varphi^I\frac{\partial}{\partial\varphi^I}\right)L = L - r_I G^I \ , \quad \bar{r}_I = r_I = \text{const} \ , \quad (3.9)$$

$$\varphi^I\frac{\partial L}{\partial\varphi^I} = \bar{\varphi}^I\frac{\partial L}{\partial\bar{\varphi}^I} \ , \quad (3.10)$$

for some real parameters r_I . These are not actually the most general conditions because one can always modify the Lagrangian by certain trivial transformations

⁵In the standard $\mathcal{N} = 2$ superspace parametrized by variables $z^A = (x^a, \theta_i^\alpha, \bar{\theta}_{\dot{\alpha}}^i)$, this transformation rotates the Grassmann variable $\theta_{\underline{1}}^\alpha$ into $\theta_{\underline{2}}^\alpha$ and vice versa.

(2.5) which distort these conditions. However, it is always possible to make the above choice. In fact, one can even set r_I to zero by a certain H -transformation (2.5b).

As a simple example of this, take the so-called improved $\mathcal{N} = 2$ tensor multiplet model [16, 39].⁶ It is described in $\mathcal{N} = 1$ superspace by the Lagrangian [16]

$$L_{\text{impr}}(G, \varphi, \bar{\varphi}) = \sqrt{G^2 + 4\varphi\bar{\varphi}} - G \ln \left(G + \sqrt{G^2 + 4\varphi\bar{\varphi}} \right). \quad (3.11)$$

This is the unique $\mathcal{N} = 2$ superconformal theory which can be constructed using a single $\mathcal{N} = 2$ tensor multiplet. Applying the first-order operator, which appears on the left of (3.9), to L_{impr} gives

$$\left(G \frac{\partial}{\partial G} + 2\varphi \frac{\partial}{\partial \varphi} \right) L_{\text{impr}} = L_{\text{impr}} - G. \quad (3.12)$$

However, one may equally well construct the same action using the Lagrangian

$$L'_{\text{impr}}(G, \varphi, \bar{\varphi}) = \sqrt{G^2 + 4\varphi\bar{\varphi}} - G \ln \left(\frac{G + \sqrt{G^2 + 4\varphi\bar{\varphi}}}{\sqrt{4\varphi\bar{\varphi}}} \right) \quad (3.13)$$

which differs only by a trivial H -transformation and indeed obeys (3.9) with $r = 0$.

Henceforth we will assume that we have modified all superconformal Lagrangians so that they obey the (weighted) homogeneity condition

$$\left(G^I \frac{\partial}{\partial G^I} + 2\varphi^I \frac{\partial}{\partial \varphi^I} \right) L = L. \quad (3.14)$$

Taking into account (3.10), this is equivalent to

$$\left(G^I \frac{\partial}{\partial G^I} + \varphi^I \frac{\partial}{\partial \varphi^I} + \bar{\varphi}^I \frac{\partial}{\partial \bar{\varphi}^I} \right) L = L. \quad (3.15)$$

Thus $L(\varphi^I, \bar{\varphi}^I, G^I)$ is a homogeneous function of first degree. If we impose also the equation (2.18), then we recover the same conditions imposed in [10].

It turns out that the above conditions on the Lagrangian guarantee invariance under the extended superconformal transformation (3.7). To prove this claim, we first point out that it is sufficient to evaluate the corresponding variation of the action for $\rho_\alpha = 0$ and $\bar{\rho}_{\dot{\alpha}} \neq 0$, since the parameters ρ_α and $\bar{\rho}_{\dot{\alpha}}$ are algebraically independent. Varying the action gives

$$\delta_{\bar{\rho}} S = \int \frac{\partial L}{\partial \varphi^I} \left\{ \bar{\rho}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} G^I + \frac{1}{2} (\bar{D}_{\dot{\alpha}} \bar{\rho}^{\dot{\alpha}}) G^I \right\} - \int \frac{\partial L}{\partial G^I} \bar{D}_{\dot{\alpha}} (\bar{\rho}^{\dot{\alpha}} \bar{\varphi}^I) \equiv I_1 + I_2, \quad (3.16)$$

⁶The improved $\mathcal{N} = 1$ tensor multiplet model was constructed in [40].

where we have denoted $\int := \int d^4x d^4\theta$. The first term may be transformed to take the form:

$$I_1 = -\frac{1}{2} \int \left\{ \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} (\bar{D}_{\dot{\alpha}} \bar{\varphi}^J) \bar{\rho}^{\dot{\alpha}} G^I + \frac{\partial^2 L}{\partial \varphi^I \partial G^J} (\bar{D}_{\dot{\alpha}} G^J) \bar{\rho}^{\dot{\alpha}} G^I - \frac{\partial L}{\partial \varphi^I} \bar{\rho}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} G^I \right\} \quad (3.17)$$

The second term in (3.16) may be transformed as

$$I_2 = \int \frac{\partial^2 L}{\partial G^I \partial \bar{\varphi}^J} (\bar{D}_{\dot{\alpha}} \bar{\varphi}^J) \bar{\rho}^{\dot{\alpha}} \bar{\varphi}^I + \int \frac{\partial^2 L}{\partial G^I \partial G^J} (\bar{D}_{\dot{\alpha}} G^J) \bar{\rho}^{\dot{\alpha}} \bar{\varphi}^I . \quad (3.18)$$

Now, let us make use of the complex conjugate of (3.14) to represent

$$\frac{\partial L}{\partial \varphi^I} = G^J \frac{\partial^2 L}{\partial G^J \partial \varphi^I} + 2 \bar{\varphi}^J \frac{\partial L}{\partial \bar{\varphi}^J \partial \varphi^I} . \quad (3.19)$$

Applying this representation to the last term in (3.17) gives

$$I_1 + I_2 = -\frac{1}{2} \int \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} (\bar{D}_{\dot{\alpha}} \bar{\varphi}^J) \bar{\rho}^{\dot{\alpha}} G^I + \int \frac{\partial^2 L}{\partial G^I \partial \bar{\varphi}^J} (\bar{D}_{\dot{\alpha}} \bar{\varphi}^J) \bar{\rho}^{\dot{\alpha}} \bar{\varphi}^I . \quad (3.20)$$

Here we have used the conditions (2.8) and (2.18).⁷ To show that $I_1 + I_2$ is equal to zero, it only remains to make use of the complex conjugate of (3.14) to obtain the identity

$$G^J \frac{\partial^2 L}{\partial G^J \partial G^I} + 2 \bar{\varphi}^J \frac{\partial L}{\partial \bar{\varphi}^J \partial G^I} = 0 . \quad (3.21)$$

The above results have a nice re-formulation in terms of the tensor-multiplet Lagrangian (2.22) which is derived using the $\mathcal{N} = 2$ projective superspace techniques. The theory is $\mathcal{N} = 2$ superconformal provided \mathcal{L} has no explicit dependence on ζ ,

$$L(\varphi^I, \bar{\varphi}^I, G^I) = \text{Re} \oint_{\gamma} \frac{d\zeta}{2\pi i \zeta} \mathcal{L}(\mathcal{G}^I(\zeta)) , \quad (3.22)$$

and is homogeneous of degree one,

$$\mathcal{G}^I \frac{\partial}{\partial \mathcal{G}^I} \mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) . \quad (3.23)$$

Given this representation, the equations (3.14) and (3.15) are satisfied identically. The $\mathcal{N} = 2$ superspace proof of superconformal invariance is given in [36].

⁷The $\mathcal{N} = 2$ supersymmetry conditions (2.8) and (2.18) are compatible with the superconformal ones, eqs. (3.10) and (3.15).

4 Self-interacting $\mathcal{N} = 2$ tensor supermultiplets: AdS supersymmetry

We are now prepared to study self-interactions of several $\mathcal{N} = 2$ tensor multiplets in AdS supersymmetry, the symmetry group being $\text{OSp}(2|4)$. In a manifestly $\mathcal{N} = 2$ supersymmetric setting, this problem has been solved in [26] (the solution will be discussed in the next section). Here we would like to address the problem using a formulation in $\mathcal{N} = 1$ AdS superspace (all essential information about this superspace is collected in Appendix A). In such an approach, an $\mathcal{N} = 2$ tensor multiplet is described by a covariantly chiral superfield φ^I , its conjugate $\bar{\varphi}^I$, and a real covariantly linear superfield G^I . The constraints are

$$\bar{\mathcal{D}}_{\dot{\alpha}}\varphi^I = 0, \quad (\bar{\mathcal{D}}^2 - 4\mu)G^I = 0, \quad \bar{G}^I = G^I, \quad I = 1, \dots, n. \quad (4.1)$$

The dynamics is described by an action of the form

$$S = \int d^4x d^4\theta E L(\varphi^I, \bar{\varphi}^I, G^I) \quad (4.2)$$

which is manifestly $\mathcal{N} = 1$ supersymmetric, i.e. invariant under arbitrary $\text{OSp}(1|4)$ transformations

$$\delta_{\xi}\varphi^I = -\xi\varphi^I, \quad \delta_{\xi}G^I = -\xi G^I, \quad \xi := \xi^A \mathcal{D}_A = \xi^a \mathcal{D}_a + \xi^{\alpha} \mathcal{D}_{\alpha} + \bar{\xi}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \quad (4.3)$$

with the AdS Killing vector field ξ^A defined by eqs. (A.17) and (A.18).

It should be pointed out that the action (4.2) does not change under arbitrary H -transformations of the Lagrangian

$$L \rightarrow L + G^I H_I(\varphi) + G^I \bar{H}_I(\bar{\varphi}), \quad (4.4)$$

with $H_I(\varphi)$ holomorphic functions.

4.1 The second supersymmetry transformation

To realize a second supersymmetry transformation, we need a real scalar ε constrained to obey the conditions [41]

$$\bar{\varepsilon} = \varepsilon, \quad (\bar{\mathcal{D}}^2 - 4\mu)\varepsilon = \bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_{\alpha}\varepsilon = 0 \implies \mathcal{D}_{\alpha\dot{\alpha}}\varepsilon = 0. \quad (4.5)$$

The parameter ε naturally originates within the $\mathcal{N} = 2$ AdS superspace approach [26], see subsection B.3 for a review. Along with ε , we will often use the chiral spinor

$$\varepsilon_{\alpha} := \mathcal{D}_{\alpha}\varepsilon, \quad \bar{\mathcal{D}}_{\dot{\alpha}}\varepsilon_{\alpha} = 0. \quad (4.6)$$

The θ -dependent parameter ε , due to the constraints (4.5), contains two components: (i) a bosonic parameter ξ which is defined by $\varepsilon|_{\theta=0} = \xi|\mu|^{-1}$ and describes the $O(2)$ rotations; and (ii) a fermionic parameter $\epsilon_\alpha := \mathcal{D}_\alpha \varepsilon|_{\theta=0}$ along with its conjugate, which generate the second supersymmetry. Schematically, the θ -expansion of ε looks like

$$\varepsilon \sim \frac{\xi}{|\mu|} + \epsilon^\alpha \theta_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} - \xi \left(\frac{\bar{\mu}}{|\mu|} \theta^2 + \frac{\mu}{|\mu|} \bar{\theta}^2 \right). \quad (4.7)$$

In accordance with the analysis given in [26], for the $\mathcal{N} = 2$ tensor multiplet in AdS the second supersymmetry transformation is

$$\delta \varphi^I = \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon G^I), \quad \delta G^I = -\mathcal{D}^\alpha(\varepsilon_\alpha \varphi^I) - \bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\varepsilon}^{\dot{\alpha}} \bar{\varphi}^I). \quad (4.8)$$

The transformation laws (4.3) and (4.8) constitute an off-shell $OSp(2|4)$ supermultiplet. Our goal is to determine those conditions which L must obey for the action (4.2) to be $\mathcal{N} = 2$ supersymmetric. Varying the action gives

$$\delta_\varepsilon S = \int d^4x d^4\theta E \left\{ \frac{1}{2} \varepsilon G^I (\bar{\mathcal{D}}^2 - 4\mu) \frac{\partial L}{\partial \varphi^I} + \varepsilon^\alpha \varphi^I \mathcal{D}_\alpha \frac{\partial L}{\partial G^I} + \text{c.c.} \right\}. \quad (4.9)$$

This can be rearranged to

$$\delta_\varepsilon S = \int d^4x d^4\theta E \left\{ \varepsilon^\alpha A_\alpha + 2\bar{\mu} \varepsilon G^I \frac{\partial L}{\partial \bar{\varphi}^I} - 4\bar{\mu} \varepsilon \varphi^I \frac{\partial L}{\partial G^I} + \text{c.c.} \right\}, \quad (4.10)$$

where

$$A_\alpha := \frac{\partial L}{\partial \bar{\varphi}^I} \mathcal{D}_\alpha G^I - \frac{\partial L}{\partial G^I} \mathcal{D}_\alpha \varphi^I. \quad (4.11)$$

The combination $\delta_\varepsilon S$ must vanish.

4.2 Derivation of conditions

An easy way to derive the conditions that L must obey is to consider the variation of $\delta_\varepsilon S$ with respect to φ^I , $\delta_\varphi \delta_\varepsilon S$. Using the properties of ε , in conjunction with the chiral reduction rule

$$\int d^4x d^4\theta E U = -\frac{1}{4} \int d^4x d^2\theta \mathcal{E} (\bar{\mathcal{D}}^2 - 4\mu) U, \quad (4.12)$$

it is clear that the variation must have the form

$$\delta_\varphi \delta_\varepsilon S = \int d^4x d^2\theta \mathcal{E} \delta \varphi^I \left\{ -\frac{1}{4} \varepsilon^\alpha (\bar{\mathcal{D}}^2 - 4\mu) \Gamma_{I\alpha} + \bar{\varepsilon}_{\dot{\alpha}} \bar{\Psi}_I^{\dot{\alpha}} + \varepsilon \Omega_I \right\}, \quad (4.13)$$

for some fields $\Gamma_{I\alpha}$, $\bar{\Psi}_I^{\dot{\alpha}}$ and Ω_I . One can show that

$$\Gamma_{I\alpha} = \left(\frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial G^I \partial G^J} \right) \mathcal{D}_\alpha G^J + \left(\frac{\partial^2 L}{\partial G^I \partial \varphi^J} - \frac{\partial^2 L}{\partial \varphi^I \partial G^J} \right) \mathcal{D}_\alpha \varphi^J. \quad (4.14)$$

This field must be such that $(\bar{\mathcal{D}}^2 - 4\mu)\Gamma_{I\alpha} = 0$. For this to occur, the Lagrangian must obey the generalized Laplace equations

$$\frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial G^I \partial G^J} = 0, \quad (4.15)$$

familiar from the Minkowski case, see section 2, as well as the condition

$$\bar{\mathcal{D}}_{\dot{\alpha}} \left(\frac{\partial^2 L}{\partial \varphi^I \partial G^J} - \frac{\partial^2 L}{\partial \varphi^J \partial G^I} \right) = 0. \quad (4.16)$$

Because L depends on G , φ , and $\bar{\varphi}$ only algebraically, this implies that

$$\frac{\partial^2 L}{\partial \varphi^I \partial G^J} - \frac{\partial^2 \mathcal{L}}{\partial \varphi^J \partial G^I} = F_{IJ}(\varphi), \quad (4.17)$$

with $F_{IJ}(\varphi)$ a closed holomorphic two-form. This is not actually an independent result; it is implied by (4.15), which can be proved as in section 2. Moreover, in complete analogy with the analysis in section 2, the two-form $F_{IJ}(\varphi)$ can be shown to be exact, and therefore it can be set to zero, $F_{IJ} = 0$, by applying an H -transformation (4.4); but this is not necessary and we will not assume it in what follows. So far the story is absolutely the same as in the rigid supersymmetric case studied in section 2. However, now comes a difference.

Assuming that L obeys (4.15), one can then show that

$$\bar{\Psi}_I^{\dot{\alpha}} = \bar{\mathcal{D}}^{\dot{\alpha}} G^J \frac{\partial R_I}{\partial \varphi_J} - \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\varphi}^J \frac{\partial R_I}{\partial G_J}, \quad (4.18)$$

where

$$\frac{1}{2} R_I := \text{Re} \left(\mu \frac{\partial L}{\partial \varphi^I} - \mu \frac{\partial^2 L}{\partial \varphi^I \partial G^J} G^J - 2\mu \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} \bar{\varphi}^J \right) \quad (4.19)$$

is a real quantity. Because $\bar{\Psi}_I^{\dot{\alpha}}$ must vanish, this implies that R_I is independent of φ and G . Since R_I is real, it must also be independent of $\bar{\varphi}$. This implies that R_I is a constant. One useful consistency check is to note that R_I is invariant under the H -transformation (4.4).

The remaining condition, the vanishing of Ω_I in (4.13), can be shown to give no new results. However, our task is not yet complete. We need one additional constraint: the constant R_I must actually be zero. To see why, consider the addition

to the Lagrangian of a term $\bar{\mu} c_I \varphi^I + \mu c_I \bar{\varphi}^I$. This obeys all the constraints we have imposed so far, and shifts R_I by a constant $2c_I \mu \bar{\mu}$. However, this is not actually $\mathcal{N} = 2$ supersymmetric, and so arbitrary values of R_I must not be allowed. Such terms have not yet been ruled out by our analysis since their $\mathcal{N} = 2$ variation depends only on G^I . We must also analyze the condition $\delta_G \delta_\varepsilon S = 0$.

Varying $\delta_\varepsilon S$ with respect to G^I leads (after a good deal of algebra) to

$$\delta_G \delta_\varepsilon S = -2 \int d^4x d^4\theta E \delta G^I R_I \varepsilon \quad (4.20)$$

after imposing the constraints we have already found. Because R_I is a constant, the integral involves just $\delta G^I \varepsilon$. Since $(\bar{\mathcal{D}}^2 - 4\mu) \mathcal{D}_\alpha \varepsilon \neq 0$, this integral does not vanish unless R_I also vanishes. Our conclusion is that

$$R_I := \frac{1}{2} \text{Re} \left(\mu \frac{\partial L}{\partial \varphi^I} - \mu \frac{\partial^2 L}{\partial \varphi^I \partial G^J} G^J - 2\mu \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} \bar{\varphi}^J \right) = 0 . \quad (4.21)$$

As compared with the situation in the rigid supersymmetric case, this is a new condition on the Lagrangian.

As an example, consider a superconformal tensor multiplet model. In accordance with the analysis in section 3, the Lagrangian $L(\varphi, \bar{\varphi}, G)$ can be taken to obey the equations

$$\left(G^I \frac{\partial}{\partial G^I} + 2\bar{\varphi}^I \frac{\partial}{\partial \bar{\varphi}^I} \right) L = L , \quad (4.22a)$$

$$\varphi^I \frac{\partial L}{\partial \varphi^I} = \bar{\varphi}^I \frac{\partial L}{\partial \bar{\varphi}^I} . \quad (4.22b)$$

It is obvious that the AdS condition (4.21) is identically satisfied.

4.3 Proof of invariance

With the conditions derived in the previous subsection, it still remains to be shown that the variation of S under the second supersymmetry transformation, eq. (4.10), vanishes. Using the definition (4.11) of the quantity A_α , which appears in (4.10), it can be checked that A_α obeys a particularly useful condition

$$\mathcal{D}_\beta A_\alpha + \mathcal{D}_\alpha A_\beta = F_{IJ} \mathcal{D}_\beta \varphi^J \mathcal{D}_\alpha \varphi^I + \bar{F}_{IJ} \mathcal{D}_\beta G^J \mathcal{D}_\alpha G^I , \quad (4.23)$$

with the holomorphic two-form $F_{IJ}(\varphi)$ defined by (4.17). As discussed earlier, we may always choose F_{IJ} to vanish, but we will take it here to be non-vanishing in the

interest of full generality. Since the two-form $F_{IJ}(\varphi)$ is exact, we may introduce a holomorphic one-form ρ_I such that

$$\partial_I \rho_J - \partial_J \rho_I = F_{IJ} . \quad (4.24)$$

Then we may introduce the combination

$$B_\alpha := A_\alpha - \frac{1}{2} \bar{F}_{IJ} G^J \mathcal{D}_\alpha G^I + \rho_I \mathcal{D}_\alpha \varphi^I \quad (4.25)$$

which obeys

$$\mathcal{D}_\beta B_\alpha + \mathcal{D}_\alpha B_\beta = 0 . \quad (4.26)$$

The variation of the action can then be written

$$\begin{aligned} \delta_\varepsilon S = \int d^4x d^4\theta E \left\{ \varepsilon^\alpha B_\alpha + \frac{1}{2} \varepsilon^\alpha (\mathcal{D}_\alpha G^I) \bar{F}_{IJ} G^J - \varepsilon^\alpha (\mathcal{D}_\alpha \varphi^I) \rho_I \right. \\ \left. + 2\bar{\mu} \varepsilon G^I \frac{\partial L}{\partial \bar{\varphi}^I} - 4\bar{\mu} \varepsilon \varphi^I \frac{\partial L}{\partial G^I} + \text{c.c.} \right\} . \end{aligned} \quad (4.27)$$

The second and third terms which we have added can be shown to vanish (the second vanishes when we write $\varepsilon^\alpha = \mathcal{D}^\alpha \varepsilon$ and integrate this spinor derivative by parts, and the third vanishes under a chiral projection). We may simplify the first term by noting that the equation (4.26) is solved by $B_\alpha = \mathcal{D}_\alpha B$ for some function $B(\varphi, \bar{\varphi}, G)$. Inserting this relation into the action and integrating by parts yields

$$\delta_\varepsilon S = -2 \int d^4x d^4\theta E \varepsilon \Omega , \quad \Omega := 2\bar{\mu} B - \bar{\mu} G^I \frac{\partial L}{\partial \bar{\varphi}^I} + 2\bar{\mu} \varphi^I \frac{\partial L}{\partial G^I} + \text{c.c.} . \quad (4.28)$$

By construction, the dependence of B on G^I and φ^I is given by

$$\begin{aligned} \frac{\partial B}{\partial G^I} &= -\frac{1}{2} \bar{F}_{IJ} G^J + \frac{\partial L}{\partial \bar{\varphi}^I} , \\ \frac{\partial B}{\partial \varphi^I} &= \rho_I - \frac{\partial L}{\partial G^I} . \end{aligned} \quad (4.29)$$

Its dependence on $\bar{\varphi}^I$ is undetermined. Using the first of these equations, we may immediately observe that

$$\frac{\partial \Omega}{\partial G^I} = R_I = 0 , \quad (4.30)$$

where R_I is defined as in (4.19). Therefore the function Ω can depend only on the variables φ and $\bar{\varphi}$. Because ε is a linear superfield, Ω may freely be modified by the transformations $\Omega \rightarrow \Omega + \Lambda + \bar{\Lambda}$, where $\Lambda = \Lambda(\varphi)$ is holomorphic, without affecting

the integral. We may interpret this as a “Kähler transformation” and so the integral $\delta_\varepsilon S$ can depend only on the “Kähler metric” constructed from Ω . However, it is easy to check that

$$\frac{\partial^2 \Omega}{\partial \varphi^I \partial \bar{\varphi}^J} = -\frac{\partial R_I}{\partial G^J} = 0 \quad (4.31)$$

and so the “Kähler metric” vanishes. Thus $\delta_\varepsilon S$ must also vanish.

4.4 Dual formulation

To construct a dual formulation of the theory, we introduce the first-order form of the action

$$S = \int d^4x d^4\theta E \left\{ L(\varphi, \bar{\varphi}, V) - V^I (\psi_I + \bar{\psi}_I) \right\}, \quad (4.32)$$

where V^I is real unconstrained and ψ_I is a covariantly chiral Lagrange multiplier. We note in passing that the original H -invariance, eq. (4.4), manifests here as

$$L \rightarrow L + V^I H_I(\varphi) + V^I \bar{H}_I(\bar{\varphi}), \quad \psi_I \rightarrow \psi_I + H_I. \quad (4.33)$$

The variables V^I can be eliminated using their equations of motion

$$\frac{\partial L}{\partial V^I} = \psi_I + \bar{\psi}_I \quad (4.34)$$

to express them in terms of the other fields. The resulting Legendre transform of L is given by

$$\mathcal{K}(\varphi^I, \bar{\varphi}^I, \psi_J + \bar{\psi}_J) = \left[L(\varphi, \bar{\varphi}, V) - V^I (\psi_I + \bar{\psi}_I) \right]_{V=V(\varphi, \bar{\varphi}, \psi+\bar{\psi})} \quad (4.35)$$

with the properties

$$\frac{\partial \mathcal{K}}{\partial \varphi^I} = \frac{\partial L}{\partial \varphi^I}, \quad \frac{\partial \mathcal{K}}{\partial \psi_I} = -V^I. \quad (4.36)$$

The dual theory has an action given purely in terms of chiral and antichiral superfields,

$$S = \int d^4x d^4\theta E \mathcal{K}(\varphi^I, \bar{\varphi}^I, \psi_J + \bar{\psi}_J), \quad (4.37)$$

which is invariant under the $\mathcal{N} = 2$ AdS isometry group, $\text{OSp}(2|4)$. The second supersymmetry transformation acts on the fields as

$$\begin{aligned} \delta \varphi^I &= -\frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu) \left(\varepsilon \frac{\partial \mathcal{K}}{\partial \psi_I} \right), \\ \delta \psi_I &= \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu) \left(\varepsilon \frac{\partial \mathcal{K}}{\partial \varphi^I} + \varepsilon F_{IJ} \frac{\partial \mathcal{K}}{\partial \psi_J} \right). \end{aligned} \quad (4.38)$$

Note the appearance of the closed two-form $F_{IJ}(\varphi)$ in the transformation rule of ψ_I .

The dual theory (4.37) is a special case of the general $\mathcal{N} = 2$ supersymmetric σ -models in AdS which will be studied in section 7, so we will delay a detailed discussion of the geometry of this model until then. For now, we only briefly mention the form that the AdS condition (4.21) takes in the dual formulation. We denote $\phi^a := (\varphi^I, \psi_I)$ as the complex coordinate of the Kähler manifold associated with the Kähler potential \mathcal{K} . The index a runs from 1 to $2n$. The AdS condition can be written

$$\mu g^{\bar{a}b} \partial_b \mathcal{K} + \bar{\mu} g^{a\bar{b}} \partial_{\bar{b}} \mathcal{K} = 2 (\mu \bar{\varphi}^I + \bar{\mu} \varphi^I) , \quad a = \bar{a} = I = 1, \dots, n . \quad (4.39)$$

Here $g^{a\bar{b}}$ is the inverse of the Kähler metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$. The equation in this form makes sense only in the particular complex coordinates singled out by the duality transformation.

4.5 Deriving the AdS condition from a superconformal tensor model

One last feature of the $\mathcal{N} = 1$ formulation of the AdS tensor model case that we would like to discuss is how to derive it from a superconformal model. Beginning with a superconformal Lagrangian L obeying the constraints

$$\left(G^I \frac{\partial}{\partial G^I} + \varphi^I \frac{\partial}{\partial \varphi^I} + \bar{\varphi}^I \frac{\partial}{\partial \bar{\varphi}^I} \right) L = L \quad (4.40a)$$

$$\varphi^I \frac{\partial L}{\partial \varphi^I} = \bar{\varphi}^I \frac{\partial L}{\partial \bar{\varphi}^I} \quad (4.40b)$$

$$\frac{\partial^2 L}{\partial G^I \partial G^J} + \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} = 0 \quad (4.40c)$$

with the index I running from 0 to n , we single out for special treatment the $I = 0$ tensor multiplet and freeze it at the values

$$\varphi^0 = i\mu , \quad \bar{\varphi}^0 = -i\bar{\mu} , \quad G^0 = 0 . \quad (4.41)$$

Starting from a frozen tensor multiplet with $\varphi^0 = \text{const}$ and $G^0 = \text{const}$, this can always be arranged by applying scale and $\text{SU}(2)$ transformations and a shadow chiral rotation. Our goal is to discover the set of isometries which keep the frozen tensor multiplet in this form and to determine what conditions the Lagrangian L obeys in terms of the unfrozen components. From now on, the index $I = 1, \dots, n$ labels only the dynamical multiplets.

We are interested in those $\mathcal{N} = 2$ superconformal transformations which keep the frozen multiplet invariant. As discussed earlier (see also Appendix B), any $\mathcal{N} = 2$ superconformal transformation decomposes into three $\mathcal{N} = 1$ transformations, and here we have to analyze only the $\mathcal{N} = 1$ superconformal and the extended superconformal transformations. Applying the $\mathcal{N} = 1$ superconformal transformation gives

$$\delta\varphi^0 = -2i\mu\sigma, \quad \delta G^0 = 0. \quad (4.42)$$

For consistency with the frozen condition (4.41), we must restrict $\sigma = 0$, and this reduces the $\mathcal{N} = 1$ superconformal Killing vector to an AdS Killing vector. The other transformation is the extended superconformal one generated by a parameter $\rho^\alpha = \mathcal{D}^\alpha \rho$ for which we find

$$\delta\varphi^0 = 0, \quad \delta G^0 = -i\mu\mathcal{D}^\alpha \rho_\alpha + i\bar{\mu}\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}}. \quad (4.43)$$

Using the constraints (B.31) and the requirement $\delta G^0 = 0$, we find $4i\bar{\mu}\mu(\bar{\rho} - \rho) = 0$ which implies that ρ must be real. This agrees with the analysis of Appendix B.3, where the second AdS supersymmetry and $O(2)$ rotation are generated by a real linear parameter ε obeying

$$(\bar{\mathcal{D}}^2 - 4\mu)\varepsilon = (\mathcal{D}^2 - 4\bar{\mu})\varepsilon = 0, \quad \mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot{\alpha}}\varepsilon = \bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha\varepsilon = 0.$$

We conclude that the surviving transformations generate the supergroup $OSp(2|4)$.

We rewrite the superconformal conditions (singling out the zero components for special treatment) as

$$\begin{aligned} \left(G^I \frac{\partial}{\partial G^I} + \varphi^I \frac{\partial}{\partial \varphi^I} + \bar{\varphi}^I \frac{\partial}{\partial \bar{\varphi}^I}\right)L &= L - i\mu \frac{\partial L}{\partial \varphi^0} + i\bar{\mu} \frac{\partial L}{\partial \bar{\varphi}^0} \\ \varphi^I \frac{\partial L}{\partial \varphi^I} - \bar{\varphi}^I \frac{\partial L}{\partial \bar{\varphi}^I} &= -i\mu \frac{\partial L}{\partial \varphi^0} - i\bar{\mu} \frac{\partial L}{\partial \bar{\varphi}^0} \\ \frac{\partial^2 L}{\partial G^I \partial G^J} + \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} &= 0 \end{aligned}$$

and make the following observation. Differentiating the first two equations with respect to $\bar{\varphi}^I$ and rearranging them, we may derive the formula

$$\frac{\partial^2 L}{\partial \varphi^0 \partial \bar{\varphi}^I} = \frac{1}{2i\mu} \left(\frac{\partial L}{\partial \bar{\varphi}^I} - G^J \frac{\partial^2 L}{\partial G^J \partial \bar{\varphi}^I} - G^0 \frac{\partial^2 L}{\partial G^0 \partial \bar{\varphi}^I} - 2\varphi^J \frac{\partial^2 L}{\partial \varphi^J \partial \bar{\varphi}^I} \right).$$

Inserting this into the relation

$$\frac{\partial^2 L}{\partial \varphi^0 \partial \bar{\varphi}^I} = \frac{\partial^2 L}{\partial \bar{\varphi}^0 \partial \varphi^I} \left(= -\frac{\partial^2 L}{\partial G^0 \partial G^I} = \text{real} \right)$$

we find

$$\bar{\mu} \left(\frac{\partial L}{\partial \bar{\varphi}^I} - G^J \frac{\partial^2 L}{\partial G^J \partial \bar{\varphi}^I} - 2\varphi^J \frac{\partial^2 L}{\partial \varphi^J \partial \bar{\varphi}^I} \right) + \text{c.c.} = 0 \quad (4.45)$$

which is exactly the extra condition (4.21) we derived for $\mathcal{N} = 2$ tensor models in AdS.

This lends credence to the following hypothesis: *all $\mathcal{N} = 2$ tensor models in AdS can be understood as superconformal tensor models with a single frozen tensor multiplet.* We will show this more explicitly (and constructively) in the next section.

5 Manifestly supersymmetric formulation

It is of interest to compare the construction of section 4 with the manifestly supersymmetric description of self-interacting $\mathcal{N} = 2$ tensor multiplets in AdS developed earlier in [26].

For general $\mathcal{N} = 2$ supersymmetric theories in AdS, the adequate superspace setting proves to be the $\mathcal{N} = 2$ AdS projective superspace $\text{AdS}^{4|8} \times \mathbb{CP}^1$ [26], which is a natural extension of the flat projective superspace $\mathbb{R}^{4|8} \times \mathbb{CP}^1$ [13, 14]. All essential information about the $\mathcal{N} = 2$ AdS superspace $\text{AdS}^{4|8}$ is collected in Appendix B. The complex projective space \mathbb{CP}^1 is conventionally parametrized by homogeneous coordinates $v^i = (v^1, v^2) \in \mathbb{C}^2 \setminus \{0\}$ defined modulo the equivalence relation $v^i \sim c v^i$. Supersymmetric matter in AdS can be described in terms of covariant projective multiplets introduced in [26] building on the off-shell formulation for general $\mathcal{N} = 2$ supergravity-matter systems developed in [27, 28]. Here we briefly recall the definition (more details can be found in [26]).

A projective supermultiplet of weight n , $\mathcal{Q}^{(n)}(\mathbf{z}, v)$, is defined to be a scalar superfield that lives on $\text{AdS}^{4|8}$, is holomorphic with respect to the isotwistor variables v^i on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

- (1) it obeys the covariant analyticity constraints

$$\mathcal{D}_\alpha^{(1)} \mathcal{Q}^{(n)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(1)} \mathcal{Q}^{(n)} = 0, \quad \mathcal{D}_\alpha^{(1)} := v_i \mathcal{D}_\alpha^i, \quad \bar{\mathcal{D}}_{\dot{\alpha}}^{(1)} := v_i \bar{\mathcal{D}}_{\dot{\alpha}}^i; \quad (5.1)$$

- (2) it is a homogeneous function of v^i of degree n ,

$$\mathcal{Q}^{(n)}(\mathbf{z}, c v) = c^n \mathcal{Q}^{(n)}(\mathbf{z}, v), \quad c \in \mathbb{C} \setminus \{0\}; \quad (5.2)$$

- (3) the $\text{OSp}(2|4)$ transformation law of $\mathcal{Q}^{(n)}$ is as follows:

$$\begin{aligned} \delta_\xi \mathcal{Q}^{(n)} &= -\left(\xi + 2\varepsilon S^{ij} J_{ij} \right) \mathcal{Q}^{(n)}, \\ S^{ij} J_{ij} \mathcal{Q}^{(n)} &:= -\left(S^{(2)} \partial^{(-2)} - n S^{(0)} \right) \mathcal{Q}^{(n)}, \end{aligned} \quad (5.3)$$

where

$$\boldsymbol{\xi} := \boldsymbol{\xi}^{\mathbf{a}} \mathcal{D}_{\mathbf{a}} + \boldsymbol{\xi}_i^{\alpha} \mathcal{D}_{\alpha}^i + \bar{\boldsymbol{\xi}}_{\dot{\alpha}}^{\dot{i}} \bar{\mathcal{D}}_{\dot{i}}^{\dot{\alpha}}$$

is an $\mathcal{N} = 2$ AdS Killing vector field, see subsection B.3, and the associated scalar parameter ε is given by eq. (B.55). In (5.3) we have introduced

$$\mathbf{S}^{(2)} := v_i v_j \mathbf{S}^{ij} , \quad \mathbf{S}^{(0)} := \frac{1}{(v, u)} v_i u_j \mathbf{S}^{ij} , \quad (5.4)$$

and also the first-order operator

$$\boldsymbol{\partial}^{(-2)} = \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i} . \quad (5.5)$$

The transformation law (5.3) involves an additional two-vector, u_i , which is only subject to the condition $(v, u) := v^i u_i \neq 0$, and is otherwise completely arbitrary. Both $\mathcal{Q}^{(n)}$ and $\delta_{\varepsilon} \mathcal{Q}^{(n)}$ are independent of u_i .

In the family of projective multiplets, one can introduce a generalized conjugation, $\mathcal{Q}^{(n)} \rightarrow \check{\mathcal{Q}}^{(n)}$, defined as

$$\check{\mathcal{Q}}^{(n)}(v) := \bar{\mathcal{Q}}^{(n)}(\bar{v} \rightarrow \check{v}) , \quad \check{v} = \mathbf{i} \sigma_2 v , \quad (5.6)$$

with $\bar{\mathcal{Q}}^{(n)}(\bar{v})$ the complex conjugate of $\mathcal{Q}^{(n)}(v)$.⁸ It is easy to check that $\check{\mathcal{Q}}^{(n)}(v)$ is a projective multiplet of weight n . One can also see that $\check{\check{\mathcal{Q}}}^{(n)} = (-1)^n \mathcal{Q}^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. The $\check{\mathcal{Q}}^{(n)}$ is called the smile-conjugate⁹ of $\mathcal{Q}^{(n)}$.

Let us also recall that there is a regular procedure to construct $\mathcal{N} = 2$ supersymmetric field theories in AdS [26]. The supersymmetric action principle is

$$S = \oint \frac{v_i dv^i}{2\pi} \int d^4x d^8\theta \mathbf{E} \frac{\mathcal{L}^{(2)}}{(\mathbf{S}^{(2)})^2} , \quad \mathbf{E}^{-1} = \text{Ber}(\mathbf{E}_{\mathcal{A}}^{\mathcal{M}}) \quad (5.7)$$

where the Lagrangian $\mathcal{L}^{(2)}(v)$ is a real weight-two projective supermultiplet constructed in terms of the dynamical projective supermultiplets.

In this section, we restrict our attention to the $\mathcal{N} = 2$ supersymmetric models of n interacting tensor multiplets, $\mathcal{G}^{I(2)}(v)$, with $I = 1, \dots, n$. Each $\mathcal{N} = 2$ tensor multiplet is a real weight-two projective multiplet of the following functional form:

$$\mathcal{G}^{I(2)}(v) = \mathcal{G}^{Iij} v_i v_j , \quad \overline{\mathcal{G}^{Iij}} = \mathcal{G}_{ij}^I = \varepsilon_{ik} \varepsilon_{jl} \mathcal{G}^{Ikl} . \quad (5.8)$$

⁸In what follows, we do not indicate explicitly the \mathbf{z} -dependence of projective supermultiplets.

⁹The smile-conjugation is the real structure pioneered by Rosly [42] and re-discovered some time later in [11, 13, 43].

A general self-coupling of $\mathcal{N} = 2$ tensor multiplets in AdS is generated by a Lagrangian of the following type:

$$\mathcal{L}_{\text{tensor}}^{(2)} = \mathcal{L}(\mathcal{G}^{I(2)}, \mathbf{S}^{(2)}) , \quad \left(\mathcal{G}^{I(2)} \frac{\partial}{\partial \mathcal{G}^{I(2)}} + \mathbf{S}^{(2)} \frac{\partial}{\partial \mathbf{S}^{(2)}} \right) \mathcal{L} = \mathcal{L} . \quad (5.9)$$

Here \mathcal{L} is an analytic homogeneous function of its arguments of degree one. In the case of superconformal tensor multiplets, the Lagrangian is independent of $\mathbf{S}^{(2)}$,

$$\frac{\partial}{\partial \mathbf{S}^{(2)}} \mathcal{L} = 0 . \quad (5.10)$$

The Lagrangian (5.9) is obtained from that describing the most general self-coupling of $n+1$ superconformal tensor multiplets by freezing one of these multiplets to coincide with $\mathbf{S}^{(2)}$.

The action (5.7) is constructed as an integral over the $\mathcal{N} = 2$ AdS superspace. It can be reduced to $\mathcal{N} = 1$ AdS superspace, $\text{AdS}^{4|4}$, according to the scheme worked out in [26]. This involves two stages. First of all, assuming (without loss of generality) that the closed integration contour in (5.7) lies outside of the north pole of \mathbb{CP}^1 , $v^i \propto (0, 1)$, all projective multiplets should be expressed in terms of the inhomogeneous complex coordinate $\zeta \in \mathbb{C}$ for \mathbb{CP}^1 which is defined as¹⁰

$$v^i = v^{\perp}(1, \zeta) . \quad (5.11)$$

In particular, associated with the Lagrangian $\mathcal{L}^{(2)}(v)$ is the superfield $\mathcal{L}(\zeta)$ defined as

$$\mathcal{L}^{(2)}(v) := i v^{\perp} v^2 \mathcal{L}(\zeta) = i (v^{\perp})^2 \zeta \mathcal{L}(\zeta) . \quad (5.12)$$

Similarly, associated with $\mathbf{S}^{(2)}(v)$ is the superfield $\mathbf{S}(\zeta)$ defined as (see eq. (B.19))

$$\mathbf{S}^{(2)}(v) := i (v^{\perp})^2 \zeta \mathbf{S}(\zeta) , \quad \mathbf{S}(\zeta) = i \left(\bar{\mu} \zeta + \mu \frac{1}{\zeta} \right) . \quad (5.13)$$

The components of \mathbf{S}^{ij} , involving the parameters μ and $\bar{\mu}$, are defined according to (B.19) and correspond to the constant torsion of $\text{AdS}^{4|4}$. Secondly, the $\mathcal{N} = 2$ superspace integral in (5.7) should be reduced to that over $\text{AdS}^{4|4}$ by making use of the analyticity conditions (5.1). Let $\mathcal{L}(\zeta)|$ denote the $\mathcal{N} = 1$ projection of the Lagrangian $\mathcal{L}(\zeta)$, see subsection B.2. Then, the $\mathcal{N} = 2$ supersymmetric action (5.7) can be shown to be equivalent to the following functional in $\text{AdS}^{4|4}$:

$$S = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta E \mathcal{L}(\zeta)| . \quad (5.14)$$

¹⁰In this chart, we can choose $u_i = (1, 0)$.

In what follows, we do not indicate the symbol of $\mathcal{N} = 1$ projection.

Given a projective supermultiplet $\mathcal{Q}^{(n)}(v)$, it can equivalently be described in terms of a properly defined superfield $\mathcal{Q}(\zeta) \propto \mathcal{Q}^{(n)}(v)$ such that the smile-conjugation $\mathcal{Q}^{(n)} \rightarrow \check{\mathcal{Q}}^{(n)}$ operates as follows:

$$\mathcal{Q}(\zeta) = \sum \mathcal{Q}_k \zeta^k \longrightarrow \check{\mathcal{Q}}(\zeta) = \sum (-1)^k \bar{\mathcal{Q}}_{-k} \zeta^k . \quad (5.15)$$

If $\mathcal{Q}(\zeta)$ is a real supermultiplet, $\check{\mathcal{Q}}(\zeta) = \mathcal{Q}(\zeta)$, then the corresponding component superfields \mathcal{Q}_k obey the reality conditions $\bar{\mathcal{Q}}_k = (-1)^k \mathcal{Q}_{-k}$. The Lagrangian $\mathcal{L}(\zeta)$ in (5.14) is real, $\check{\mathcal{L}}(\zeta) = \mathcal{L}(\zeta)$, which implies the action (5.14) is real.

In the case of $\mathcal{N} = 2$ tensor multiplets, we represent

$$\mathcal{G}^{I(2)}(v) := i(v^\perp)^2 \zeta \mathcal{G}^I(\zeta) , \quad \mathcal{G}^I(\zeta) = \frac{1}{\zeta} \varphi^I + G^I - \zeta \bar{\varphi}^I , \quad \check{\mathcal{G}}^I(\zeta) = \mathcal{G}^I(\zeta) . \quad (5.16)$$

The analyticity conditions (5.1) on $\mathcal{G}^{I(2)}(v)$ can be shown to imply that the $\mathcal{N} = 1$ scalar superfields φ^I and G^I obey the constraints

$$\bar{\mathcal{D}}_{\dot{\alpha}} \varphi^I = 0 , \quad (\bar{\mathcal{D}}^2 - 4\mu) G^I = 0 , \quad \bar{G}^I = G^I . \quad (5.17)$$

Reformulated in $\mathcal{N} = 1$ AdS superspace, the tensor multiplet model generated by (5.9) becomes

$$S_{\text{tensor}} = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta E \mathcal{L}(\mathcal{G}^I(\zeta), \mathcal{S}(\zeta)) . \quad (5.18)$$

Upon evaluation of the contour integral, this action takes the form (4.2) with

$$L(\varphi^I, \bar{\varphi}^I, G^I) = \oint \frac{d\zeta}{2\pi i \zeta} \mathcal{L}(\mathcal{G}^I(\zeta), i\bar{\mu} \zeta + i\mu \zeta^{-1}) . \quad (5.19)$$

In contrast to the rigid supersymmetric case, see subsection 2.2, the integrand on the right is not allowed to depend on ζ in an arbitrary way, but only through the real combination $i(\bar{\mu} \zeta + \mu \zeta^{-1})$.

One way of understanding this ζ dependence is to investigate how the condition (4.21) is satisfied by the integral (5.19). We begin by observing that the quantity R_I (4.19) can be written

$$R_I = \oint \frac{d\zeta}{2\pi i \zeta} \left(\frac{\mu}{\zeta} \frac{\partial \mathcal{L}}{\partial \mathcal{G}^I} - \frac{\mu}{\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathcal{G}^J} G^J + 2\mu \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathcal{G}^J} \bar{\varphi}^J \right) + \text{c.c.} \quad (5.20)$$

The first term can be rewritten (neglecting a total contour derivative) as

$$R_I = \oint \frac{d\zeta}{2\pi i \zeta} \left(\mu \frac{d}{d\zeta} \frac{\partial \mathcal{L}}{\partial \mathcal{G}^I} - \frac{\mu}{\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathcal{G}^J} G^J + 2\mu \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathcal{G}^J} \bar{\varphi}^J \right) + \text{c.c.} \quad (5.21)$$

Now if we make use of the fact that the ζ dependence of \mathcal{L} occurs implicitly in the superfields \mathcal{G}^I and explicitly in the combination $\mathbf{S}(\zeta)$ (5.13), this integral may be further simplified to

$$R_I = \oint \frac{d\zeta}{2\pi i \zeta} \left(\mu \frac{d\mathbf{S}}{d\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathbf{S}} - \frac{\mu}{\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathcal{G}^J} \mathcal{G}^J \right) + \text{c.c.} \quad (5.22)$$

Because the $\mathcal{N} = 2$ Lagrangian \mathcal{L} is homogeneous of degree one in terms of \mathcal{G}^I and \mathbf{S} (5.9), we have

$$\begin{aligned} R_I &= \oint \frac{d\zeta}{2\pi i \zeta} \left(\mu \frac{d\mathbf{S}}{d\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathbf{S}} + \frac{\mu}{\zeta} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathbf{S}} \mathbf{S} \right) + \text{c.c.} \\ &= \oint \frac{d\zeta}{2\pi i \zeta} \left(2i\mu\bar{\mu} \frac{\partial^2 \mathcal{L}}{\partial \mathcal{G}^I \partial \mathbf{S}} \right) + \text{c.c.} = 0 \end{aligned} \quad (5.23)$$

after making use of the explicit form (5.13) of $\mathbf{S}(\zeta)$.

6 $\mathcal{N} = 2$ supersymmetric σ -models in AdS

$\mathcal{N} = 2$ supersymmetric σ -models in AdS can be formulated using off-shell hypermultiplets in $\mathcal{N} = 2$ AdS superspace [26]. The virtue of the approach pursued in [26] is that the superfield Lagrangian can have an arbitrary functional form. The technical disadvantage of this approach is that, upon reduction to $\mathcal{N} = 1$ AdS superspace (elaborated in [26]), the action functional involves not only physical superfields (chiral and complex linear ones) but also an infinite set of auxiliary fields which have to be eliminated using their nonlinear algebraic equations of motion. Here we will pursue a complementary approach, in the spirit of [15], by formulating the action in terms of the physical $\mathcal{N} = 1$ chiral superfields only and then determining conditions on the Lagrangian for the theory to possess $\mathcal{N} = 2$ supersymmetry. The main results of this work were previously reported in a brief letter [44]. We refer the reader there for details of the component form of the action and the supersymmetry transformation rules.

6.1 $\mathcal{N} = 1$ supersymmetric σ -models in AdS

The most general nonlinear σ -model in $\mathcal{N} = 1$ AdS superspace is given by

$$S = \int d^4x d^4\theta E \mathcal{K}(\phi^a, \bar{\phi}^{\bar{b}}) . \quad (6.1)$$

The dynamical variables ϕ^a are covariantly chiral superfields, $\bar{\mathcal{D}}_{\dot{\alpha}}\phi^a = 0$, and at the same time local complex coordinates of a complex manifold \mathcal{M} . Unlike in the Minkowski case, the action does not possess Kähler invariance since

$$\int d^4x d^4\theta E F(\phi^a) = \int d^4x d^2\theta \mathcal{E} \mu F(\phi^a) \neq 0 , \quad (6.2)$$

with \mathcal{E} the chiral density. Nevertheless, Kähler invariance naturally emerges if we represent the Lagrangian as

$$\mathcal{K}(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) + \frac{1}{\mu} W(\phi) + \frac{1}{\bar{\mu}} \bar{W}(\bar{\phi}) , \quad (6.3)$$

for some Kähler potential K and superpotential W . Under a Kähler transformation, these transform as

$$K \rightarrow K + F + \bar{F}, \quad W \rightarrow W - \mu F . \quad (6.4)$$

The Kähler metric defined by

$$g_{a\bar{b}} := \partial_a \partial_{\bar{b}} \mathcal{K} = \partial_a \partial_{\bar{b}} K \quad (6.5)$$

is obviously invariant under (6.4).

The nonlinear σ -model (6.1) is manifestly invariant under arbitrary $\mathcal{N} = 1$ AdS isometry transformations

$$\delta_{\xi} \phi^a = -\xi \phi^a , \quad \xi := \xi^A \mathcal{D}_A = \xi^a \mathcal{D}_a + \xi^{\alpha} \mathcal{D}_{\alpha} + \bar{\xi}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \quad (6.6)$$

with the AdS Killing vector field ξ^A defined by eqs. (A.17) and (A.18).

Because of (6.2), the Lagrangian \mathcal{K} in (6.1) should be a globally defined function on the Kähler target space \mathcal{M} . This immediately implies that the Kähler two-form, $\Omega = 2i g_{a\bar{b}} d\phi^a \wedge d\bar{\phi}^{\bar{b}}$, associated with (6.5), is exact and hence \mathcal{M} is necessarily non-compact. We see that the σ -model couplings in AdS are more restrictive than in the Minkowski case. The same conclusion follows from our earlier analysis of AdS supercurrent multiplets [19]. In [19] we demonstrated that $\mathcal{N} = 1$ AdS supersymmetry allows the existence of just one minimal $(12 + 12)$ supercurrent, unlike the case of Poincaré supersymmetry which admits three $(12 + 12)$ supercurrents. The corresponding AdS supercurrent is associated with the old minimal supergravity and coincides with the AdS extension of the Ferrara-Zumino multiplet [45]. An immediate application of this result is that all supersymmetric σ -models in AdS must possess a well-defined Ferrara-Zumino multiplet. The same conclusion also follows from the exactness of Ω and earlier results of Komargodski and Seiberg [46] who demonstrated

that all *rigid* supersymmetric σ -models with an exact Kähler two-form possess a well-defined Ferrara-Zumino multiplet. The exactness of Ω for the general $\mathcal{N} = 1$ σ -models in AdS was independently observed in [20] and [21] which appeared shortly after [19].

We should discuss briefly how the structure (6.1) emerges within a supergravity description (see also [20]). Recall that nonlinear σ -models may be coupled to supergravity via

$$S = -\frac{3}{\kappa^2} \int d^4x d^4\theta E e^{-\kappa^2 K/3} + \int d^4x d^2\theta \mathcal{E} W_{\text{sugra}} + \int d^4x d^2\bar{\theta} \bar{\mathcal{E}} \bar{W}_{\text{sugra}} \quad (6.7)$$

where the Kähler potential K and the superpotential W_{sugra} transform under Kähler transformations as

$$K \rightarrow K + F + \bar{F}, \quad W_{\text{sugra}} \rightarrow e^{-\kappa^2 F} W_{\text{sugra}} . \quad (6.8)$$

The parameter κ corresponds to the inverse Planck mass which we will take to be vanishingly small to freeze out the gravitational dynamics. The cleanest way to derive an AdS model from this supergravity model is to assume W_{sugra} is dominated by a cosmological term with a (relatively) small correction associated with the AdS superpotential,

$$W_{\text{sugra}} = \frac{\mu}{\kappa^2} + W + \mathcal{O}(\kappa^2) .$$

The precise choice of the $\mathcal{O}(\kappa^2)$ corrections is irrelevant once the small κ limit is taken, but the *cleanest* choice is to choose

$$W_{\text{sugra}} = \frac{\mu}{\kappa^2} \exp\left(\frac{\kappa^2}{\mu} W\right) . \quad (6.9)$$

For this choice, the AdS Kähler transformation (6.4) matches the supergravity Kähler transformation (6.8). The terms which diverge in a small κ limit correspond to pure supergravity with a cosmological constant and the supergravity equations of motion may be solved to yield an AdS solution, freezing the supergravity structure. The terms which remain as κ tends to zero can be shown to take the form (6.1) with \mathcal{K} given by (6.3).

6.2 The second supersymmetry transformation

Next we look for those restrictions on the target space geometry which guarantee that the action (6.1) is $\mathcal{N} = 2$ supersymmetric. We make the following ansatz for the action of a second supersymmetry on the chiral superfield ϕ^a :

$$\delta_\varepsilon \phi^a = \frac{1}{2} (\bar{\mathcal{D}}^2 - 4\mu) (\varepsilon \bar{\Omega}^a) , \quad (6.10)$$

where $\bar{\Omega}^a$ is a function of ϕ and $\bar{\phi}$ which has to be determined.

The transformation law (6.10) is a generalization of that derived in [26] in the case of a free off-shell $\mathcal{N} = 2$ hypermultiplet $\phi^a = (\Phi, \Psi)$ described by the action

$$S = \int d^4x d^4\theta E \left(\bar{\Phi}\Phi + \bar{\Psi}\Psi + i\frac{m}{\mu}\Psi\Phi - i\frac{m}{\bar{\mu}}\bar{\Psi}\bar{\Phi} \right), \quad (6.11)$$

with m a mass parameter.¹¹ This action is invariant under the second supersymmetry transformation¹²

$$\delta_\varepsilon \Phi = \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\bar{\Psi}), \quad \delta_\varepsilon \Psi = -\frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\bar{\Phi}). \quad (6.12)$$

The ansatz (6.10) also has a correct super-Poincaré limit [15] (see also [17]).

On the mass shell, the right-hand side of (6.10) should transform as a vector field of type (1, 0) under reparametrizations of the target space. Due to the constraints (4.5), the transformation $\delta\phi^a$ may be rewritten

$$\delta_\varepsilon \phi^a = \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Omega}^a + \frac{1}{2} \varepsilon \bar{\mathcal{D}}^2 \bar{\Omega}^a. \quad (6.13)$$

This makes clear that $\bar{\Omega}^a$ is defined only up to a holomorphic vector,

$$\bar{\Omega}^a \rightarrow \bar{\Omega}^a + H^a(\phi). \quad (6.14)$$

6.3 Deriving the conditions of invariance

Let us derive the conditions on \mathcal{K} and $\bar{\Omega}^a$ so that the variation of the action

$$\delta_\varepsilon S = \int d^4x d^4\theta E \left\{ \frac{1}{2} \mathcal{K}_b (\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\bar{\Omega}^b) + \frac{1}{2} \mathcal{K}_{\bar{b}} (\mathcal{D}^2 - 4\bar{\mu})(\varepsilon\bar{\Omega}^{\bar{b}}) \right\} \quad (6.15)$$

is zero. An easy way to find a set of *necessary* conditions is to require not δS itself to vanish but rather its variation under arbitrary deformations of a chiral field ϕ^a :

$$\begin{aligned} \delta_\phi \delta_\varepsilon S = & -\frac{1}{8} \int d^4x d^2\theta \mathcal{E} \delta\phi^a (\bar{\mathcal{D}}^2 - 4\mu) \left\{ \mathcal{K}_{ab} (\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\bar{\Omega}^b) + \varepsilon\bar{\Omega}^b{}_{,a} (\bar{\mathcal{D}}^2 - 4\mu) \mathcal{K}_b \right. \\ & \left. + g_{a\bar{b}} (\mathcal{D}^2 - 4\bar{\mu})(\varepsilon\bar{\Omega}^{\bar{b}}) + \varepsilon\bar{\Omega}^{\bar{b}}{}_{,a} (\mathcal{D}^2 - 4\bar{\mu}) \mathcal{K}_{\bar{b}} \right\}. \end{aligned} \quad (6.16)$$

Certainly if (6.15) vanishes, then so should this quantity. It turns out that ensuring the vanishing of (6.16) gives us all the constraints required to show that (6.15) vanishes. Let us work these out now.

¹¹The choice $m = 0$ corresponds to the superconformal massless hypermultiplet.

¹²In conjunction with (6.6), the transformation law (6.12) defines an off-shell (Fayet-Sohnius-type) hypermultiplet in AdS.

We begin by looking for all terms which will yield ε_α under the chiral projection. There is only one, and it is found in the third term of (6.16)

$$-\frac{1}{8}(\bar{\mathcal{D}}^2 - 4\mu)\left\{g_{a\bar{b}}(\mathcal{D}^2 - 4\bar{\mu})(\varepsilon\Omega^{\bar{b}})\right\} = -\frac{1}{4}\varepsilon^\alpha(\bar{\mathcal{D}}^2 - 4\mu)\left\{g_{a\bar{b}}\Omega^{\bar{b}}{}_{,c}\mathcal{D}_\alpha\phi^c\right\} + \dots \quad (6.17)$$

where we have made use of the chirality of ε_α and neglected in the ellipsis all terms involving ε and $\bar{\varepsilon}_{\dot{\alpha}}$. The term in braces must vanish; this implies that the quantity

$$\omega_{ab} := g_{a\bar{c}}\Omega^{\bar{c}}{}_{,b} \quad (6.18)$$

must be holomorphic,

$$\omega_{ab} = \omega_{ab}(\phi) \quad \Longleftrightarrow \quad \nabla_{\bar{c}}\omega_{ab} = 0. \quad (6.19)$$

Next we consider all the other contributions from the third and fourth terms of (6.16). These amount to

$$\begin{aligned} & -\frac{1}{8}(\bar{\mathcal{D}}^2 - 4\mu)\left\{\varepsilon g_{a\bar{b}}\mathcal{D}^2\Omega^{\bar{b}} + \varepsilon\mathcal{D}^2\mathcal{K}_{\bar{b}}\Omega^{\bar{b}}{}_{,a} - 4\varepsilon\bar{\mu}\mathcal{K}_{\bar{b}}\Omega^{\bar{b}}{}_{,a}\right\} \\ & = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 4\mu)\left\{\varepsilon\mathcal{D}^2\mathcal{K}_{\bar{b}}g^{\bar{b}c}(\omega_{ac} + \omega_{ca}) + \varepsilon\mathcal{D}^\alpha\phi^c\mathcal{D}_\alpha\phi^d\nabla_d\omega_{ac}\right\}. \end{aligned} \quad (6.20)$$

When chirally projected, these are the only terms which will give contributions proportional to $\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}}\phi^c$ and $\mathcal{D}_{\alpha\dot{\alpha}}\phi^c\mathcal{D}^{\dot{\alpha}\alpha}\phi^d$ respectively, so we must require both of these to vanish. This leads to the conditions

$$\omega_{ab} = -\omega_{ba}, \quad \nabla_c\omega_{ab} = 0, \quad (6.21)$$

which tells us that ω_{ab} is indeed a covariantly constant holomorphic two-form. These conditions completely eliminate the third and fourth terms of (6.16).

Finally, we must ensure cancellation of the first and second terms of (6.16). Making use of the identity

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4\mu)\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = 0 \quad (6.22)$$

for arbitrary $\bar{\psi}^{\dot{\alpha}}$, we may rearrange the first and second terms to

$$\frac{1}{8}(\bar{\mathcal{D}}^2 - 4\mu)\left\{\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{b}}\nabla_a\bar{\Omega}_{\bar{b}} + 4\varepsilon\mu\partial_a(\bar{\Omega}^{\bar{b}}\mathcal{K}_{\bar{b}}) + 4\varepsilon\bar{\mu}\mathcal{K}_{\bar{b}}g^{\bar{b}b}\omega_{ba}\right\} \quad (6.23)$$

where we have defined $\bar{\Omega}_{\bar{b}} := g_{\bar{b}c}\bar{\Omega}^c$ and made use of the anti-holomorphy of $\omega_{a\bar{b}}$ to eliminate an extraneous term. We must apply the chiral projection operator and check that the coefficients of $\bar{\varepsilon}_{\dot{\alpha}}$ and ε vanish separately.

At this point it is useful to observe that so far we have established the same set of constraints (6.18) and (6.21) as imposed in the globally supersymmetric case [15]. It follows that only terms which explicitly depend on μ (or $\bar{\mu}$) after the chiral projection will need to be checked. Taking the chiral projection and selecting out just the terms involving μ (or $\bar{\mu}$) and $\bar{\varepsilon}_{\dot{\alpha}}$, we find

$$\left\{ \mu \partial_a (\mathcal{K}_b g^{b\bar{b}} \bar{\omega}_{\bar{b}\bar{c}}) + \bar{\mu} \partial_{\bar{c}} (\mathcal{K}_{\bar{b}} g^{\bar{b}b} \omega_{ba}) \right\} \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{c}} . \quad (6.24)$$

The term in braces must vanish. Defining

$$V^a := \frac{\mu}{2|\mu|} \omega^{ab} \mathcal{K}_b , \quad V^{\bar{a}} := \frac{\bar{\mu}}{2|\mu|} \bar{\omega}^{\bar{a}\bar{b}} \mathcal{K}_{\bar{b}} \quad (6.25)$$

we find the cancellation condition amounts to

$$\nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0 . \quad (6.26)$$

In addition, we observe that by construction

$$\nabla_a V_b = -\frac{\bar{\mu}}{2|\mu|} \omega_{ab} , \quad (6.27)$$

which leads to

$$\nabla_a V_b + \nabla_b V_a = 0 . \quad (6.28)$$

There still remain several terms in $\delta_{\phi} \delta_{\varepsilon} S$ which we have not yet analyzed. However, their total contribution can be shown to vanish by using the conditions we have already established, and thus $\delta_{\phi} \delta_{\varepsilon} S = 0$. These conditions are:

(i) the existence of a covariantly constant two-form

$$\omega_{ab} := g_{a\bar{b}} \Omega^{\bar{b}}_{,b} = -\omega_{ba} , \quad \nabla_c \omega_{ab} = \nabla_{\bar{c}} \omega_{ab} = 0 \quad \implies \quad \omega_{ab} = \omega_{ab}(\phi) ; \quad (6.29)$$

(ii) the existence of a certain Killing vector field obeying

$$V^a := \frac{\mu}{2|\mu|} \omega^{ab} \mathcal{K}_b, \quad \nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0 , \quad \nabla_a V_b = -\frac{\bar{\mu}}{2|\mu|} \omega_{ab} = -\nabla_b V_a . \quad (6.30)$$

The first condition occurs both in the Minkowski and AdS cases. The second condition is characteristic of AdS supersymmetry only.

It should be remarked that, modulo transformations (6.14), we can choose

$$\bar{\Omega}^a(\phi, \bar{\phi}) = \omega^{ab}(\phi) \mathcal{K}_b(\phi, \bar{\phi}) , \quad (6.31)$$

similarly to the super-Poincaré case [15]. The specific feature of the AdS case is that \mathcal{K}_b is a (globally-defined) one-form, and thus $\bar{\Omega}^a$ is necessarily a vector field. Comparing the expression for $\bar{\Omega}^a$ with (6.25) shows that $V^a = \mu \bar{\Omega}^a / 2|\mu|$. The choice (6.31) will be assumed in what follows.

There is an important piece of information encoded in the first relation in (6.29) which leads to

$$2\omega_{ab}(\phi) = \nabla_b \Omega_a - \nabla_a \Omega_b = \partial_b \Omega_a(\phi, \bar{\phi}) - \partial_a \Omega_b(\phi, \bar{\phi}) . \quad (6.32)$$

(It should be recalled that we have made the choice $\Omega_a = g_{a\bar{b}} \Omega^{\bar{b}}$.) The left-hand side of (6.32) does not depend on $\bar{\phi}$, which makes it possible to evaluate the right-hand side locally by giving these variables any given values $\bar{\phi}_0$, that is

$$\omega_{ab}(\phi) = \partial_a \rho_b(\phi) - \partial_b \rho_a(\phi) , \quad \rho_a(\phi) := -\frac{1}{2} \Omega_a(\phi, \bar{\phi}_0) . \quad (6.33)$$

This gives an explicit local expression for ρ_a .¹³

6.4 Proof of invariance

Having derived the conditions (6.29) and (6.30), we must still show that the action is indeed invariant. We begin by noting that the variation of the action $\delta_\epsilon S$ may be written

$$\delta_\epsilon S = -\frac{1}{2} \int d^4x d^4\theta E \left(\varepsilon^\alpha A_\alpha + \bar{\varepsilon}_{\dot{\alpha}} \bar{A}^{\dot{\alpha}} \right) , \quad A_\alpha := \mathcal{D}_\alpha \phi^b \omega_{bc} g^{c\bar{c}} \mathcal{K}_{\bar{c}} . \quad (6.34)$$

From this point we may proceed in a manner quite analogous to the $\mathcal{N} = 2$ tensor model considered in subsection 4.3.

We first observe that the quantity A_α obeys the condition

$$\mathcal{D}_\beta A_\alpha + \mathcal{D}_\alpha A_\beta = -2\mathcal{D}_\beta \phi^b \mathcal{D}_\alpha \phi^c \omega_{bc} . \quad (6.35)$$

The form $\omega_{ab}(\phi)$ is locally exact and given by (6.33). This representation is crucial for the proof below. We include a more direct proof of invariance, which does not make use of (6.33), in Appendix C.

Because ρ_a is holomorphic, it is possible to add trivial terms to the integrand,

$$\delta_\epsilon S = -\frac{1}{2} \int d^4x d^4\theta E \left\{ \varepsilon^\alpha (A_\alpha + 2\mathcal{D}_\alpha \phi^b \rho_b) + \bar{\varepsilon}_{\dot{\alpha}} (\bar{A}^{\dot{\alpha}} + 2\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{b}} \bar{\rho}_{\bar{b}}) \right\} . \quad (6.36)$$

¹³A similar trick was used by Bagger and Xiong in [17] in a slightly different context.

These additional terms contribute nothing since the chiral projection of $\varepsilon^\alpha \mathcal{D}_\alpha \phi^b \rho_b$ vanishes using the chirality of ε^α and ρ_b . Defining

$$B_\alpha := A_\alpha + 2\mathcal{D}_\alpha \phi^b \rho_b = \mathcal{D}_\alpha \phi^b \left(\omega_{bc} g^{c\bar{c}} \mathcal{K}_{\bar{c}} + 2\rho_b \right) , \quad (6.37)$$

we observe that $\mathcal{D}_\beta B_\alpha + \mathcal{D}_\alpha B_\beta = 0$. This equation is solved by

$$B_\alpha = \mathcal{D}_\alpha B \quad (6.38)$$

for some function $B(\phi, \bar{\phi})$ on the target space. Not much may be said about B in the general case except that

$$\partial_b B = \omega_{bc} g^{c\bar{c}} \mathcal{K}_{\bar{c}} + 2\rho_b \quad (6.39)$$

which follows from (6.37) and (6.38). The formula (6.36) may be then integrated by parts to yield

$$\delta_\varepsilon S = 2 \int d^4x d^4\theta E \varepsilon (\bar{\mu} B + \mu \bar{B}) . \quad (6.40)$$

Our task is essentially complete. Since ε is a real linear superfield, the quantity resembles the integrand $\int d^4x d^4\theta E L K$ which is well-known to depend only on the value of the Kähler metric constructed from K when L is a linear superfield. In (6.40), ε is indeed a linear superfield while $\bar{\mu} B + \mu \bar{B}$ plays the role of a “Kähler potential” in this analogy. Its corresponding Kähler metric,

$$\begin{aligned} \partial_a \partial_{\bar{b}} (\bar{\mu} B + \mu \bar{B}) &= \bar{\mu} \partial_{\bar{b}} (\omega_{ac} g^{c\bar{c}} \mathcal{K}_{\bar{c}} + 2\rho_a) + \mu \partial_a (\omega_{\bar{b}\bar{c}} g^{\bar{c}c} \mathcal{K}_c + 2\bar{\rho}_{\bar{b}}) \\ &= \bar{\mu} \partial_{\bar{b}} (\omega_{ac} g^{c\bar{c}} \mathcal{K}_{\bar{c}}) + \mu \partial_a (\omega_{\bar{b}\bar{c}} g^{\bar{c}c} \mathcal{K}_c) , \end{aligned} \quad (6.41)$$

vanishes due to Killing vector condition (6.30). This implies that (6.40) is indeed equal to zero.

6.5 Closure of the supersymmetry algebra

Let us calculate the commutator of two second supersymmetry transformations (6.10). This calculation is rather short and the result is

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}] \phi^a = \omega^{ac} \omega_{cb} \left(-\frac{1}{2} \tilde{\xi}^{\alpha\dot{\alpha}} \mathcal{D}_{\alpha\dot{\alpha}} + \tilde{\xi}^\alpha \mathcal{D}_\alpha \right) \phi^b , \quad (6.42)$$

where

$$\tilde{\xi}^{\alpha\dot{\alpha}} := 4i(\varepsilon_1^\alpha \bar{\varepsilon}_2^{\dot{\alpha}} - \varepsilon_2^\alpha \bar{\varepsilon}_1^{\dot{\alpha}}) , \quad \tilde{\xi}^\alpha := 2\mu(\varepsilon_2^\alpha \varepsilon_1 - \varepsilon_1^\alpha \varepsilon_2) \quad (6.43)$$

are the components of the first-order operator $\xi_{[\varepsilon_2, \varepsilon_1]} = -\frac{1}{2}\tilde{\xi}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} + \tilde{\xi}^{\alpha}\mathcal{D}_{\alpha} + \tilde{\bar{\xi}}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}$ which proves to be an AdS Killing vector field, see eqs. (A.17) and (A.18). If we impose

$$\omega^{ac}\omega_{cb} = -\delta^a_b, \quad (6.44)$$

then the above result turns into

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\phi^a = -\xi_{[\varepsilon_2, \varepsilon_1]}\phi^a. \quad (6.45)$$

We see from (6.45) that the commutator $[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\phi^a$ closes off the mass shell. This is similar to the supersymmetry structure within the Bagger-Xiong formulation [17] for $\mathcal{N} = 2$ rigid supersymmetric σ -models. However, in the case of flat superspace, the commutator of the first and the second supersymmetries closes *only* on-shell [17]. What about the AdS case? Computing the commutator of the $\mathcal{N} = 1$ AdS transformation and the second supersymmetry transformation gives

$$[\delta_{\xi}, \delta_{\varepsilon}]\phi^a = -\frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\left((\xi\varepsilon)\bar{\Omega}^a\right). \quad (6.46)$$

Since ξ is an $\mathcal{N} = 1$ Killing vector field, the parameter $\varepsilon' = \xi\varepsilon$ obeys the constraints (4.5) and hence generates a second supersymmetry transformation. We observe that commuting the $\mathcal{N} = 1$ AdS transformation and the second supersymmetry gives a second supersymmetry transformation,

$$[\delta_{\xi}, \delta_{\varepsilon}]\phi^a = -\delta_{\xi\varepsilon}\phi^a. \quad (6.47)$$

As a result, the algebra of $\text{OSp}(2|4)$ transformations is closed off the mass shell!¹⁴

Let us return to the equation (6.44). Its implications are the same as in the super-Poincaré case [15]. In addition to the canonical complex structure

$$J_3 = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix}, \quad (6.48)$$

we may construct two more using $\omega^{a_{\bar{b}}}$

$$J_1 = \begin{pmatrix} 0 & \omega^{a_{\bar{b}}} \\ \omega^{\bar{a}}_b & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i\omega^{a_{\bar{b}}} \\ -i\omega^{\bar{a}}_b & 0 \end{pmatrix} \quad (6.49)$$

¹⁴It should be mentioned that the linearized action for *all* massless supermultiplets of arbitrary superspin in $\mathcal{N} = 1$ AdS superspace [41] is also invariant under $\mathcal{N} = 2$ supersymmetry transformations which close off-shell.

such that \mathcal{M} is Kähler with respect to each of them. The operators $J_A = (J_1, J_2, J_3)$ obey the quaternionic algebra

$$J_A J_B = -\delta_{AB} \mathbb{I} + \epsilon_{ABC} J_C . \quad (6.50)$$

Thus, \mathcal{M} is a hyperkähler manifold. In accordance with the discussion in subsection 6.1, this manifold is non-compact. The above analysis also shows that \mathcal{M} must possess a special Killing vector.

Using (6.44), it is easy to establish the equivalence

$$(\bar{\mathcal{D}}^2 - 4\mu)\mathcal{K}_a = 0 \quad \Longleftrightarrow \quad (\bar{\mathcal{D}}^2 - 4\mu)(\omega^{ab}\mathcal{K}_b) = 0 . \quad (6.51)$$

This results implies that the following rigid symmetry of the $\mathcal{N} = 2$ σ -model

$$\delta\phi^a = \zeta(\bar{\mathcal{D}}^2 - 4\mu)(\omega^{ab}\mathcal{K}_b) , \quad \zeta \in \mathbb{C} \quad (6.52)$$

is trivial.

It is well-known that when $\mathcal{N} = 2$ σ -models are coupled to supergravity, their target spaces must be quaternionic Kähler manifolds [47]. Unlike the hyperkähler spaces which are Ricci-flat, their quaternionic Kähler cousins are Einstein spaces with a non-zero constant scalar curvature (see, e.g., [48] for a review). Since AdS is a curved geometry, one may wonder whether the target spaces of $\mathcal{N} = 2$ σ -models in AdS should also be quaternionic Kähler. Yet we have shown here that within AdS, the geometry is hyperkähler just as in Minkowski space. The reason is simple. As shown in [47], the scalar curvature in the target space of locally supersymmetric σ -models must be nonzero and proportional to κ^2 ,

$$R = -8\kappa^2(n^2 + 2n) , \quad (6.53)$$

where the real dimension of the target space is $4n$. But AdS (or Minkowski) space can be interpreted as the $\kappa^2 \rightarrow 0$ limit of supergravity with (or without) a cosmological constant μ . In such a limit, we find indeed that the quaternionic Kähler requirement from supergravity reduces to a hyperkähler requirement.

7 Geometric aspects of $\mathcal{N} = 2$ σ -models in AdS

In this section we would like to take a closer look at the geometric properties of the Killing vector field (6.25) which is characteristic of the target space of any $\mathcal{N} = 2$ supersymmetric σ -model in AdS. For that it is useful to recall the key facts about (tri-)holomorphic (Killing) vector fields.

7.1 Holomorphic (Killing) vector fields

Consider a Kähler manifold $(\mathcal{M}, g_{\mu\nu}, J^\mu{}_\nu)$, with $g_{\mu\nu}$ the Kähler metric and $J^\mu{}_\nu$ the complex structure, which obeys $\nabla_\lambda J^\mu{}_\nu = 0$. A vector field $V = V^\mu \partial_\mu$ is said to be holomorphic with respect to J if

$$\mathcal{L}_V J = -J^\rho{}_\nu \nabla_\rho V^\mu + J^\mu{}_\rho \nabla_\nu V^\rho = 0 . \quad (7.1)$$

If in addition V is a Killing vector field,

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 , \quad (7.2)$$

then V proves to be a Hamiltonian vector field with respect to the symplectic two-form $\mathcal{J} = J_{\mu\nu} d\phi^\mu \wedge d\phi^\nu \equiv g_{\mu\lambda} J^\lambda{}_\nu d\phi^\mu \wedge d\phi^\nu$, that is

$$d(i_V \mathcal{J}) = 0 \quad \Longleftrightarrow \quad \mathcal{L}_V \mathcal{J} = 0 . \quad (7.3)$$

Any two of the conditions (7.1), (7.2) and (7.3) imply the third one [43].

We choose local complex coordinates, $\phi^\mu = (\phi^a, \bar{\phi}^{\bar{a}})$, such that the complex structure becomes diagonal, and the Kähler metric takes the form $ds^2 = 2g_{a\bar{b}} d\phi^a d\bar{\phi}^{\bar{b}}$. Then, the holomorphy condition (7.1) imposed on our vector field

$$V = V^a \frac{\partial}{\partial \phi^a} + \bar{V}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \quad (7.4)$$

amounts to the requirement that V^a is independent of $\bar{\phi}$, $V^a = V^a(\phi)$. If V is also a Killing vector, then the condition (7.3) is equivalent to

$$V_a := g_{a\bar{b}} \bar{V}^{\bar{b}} = i \partial_a \Upsilon , \quad \bar{\Upsilon} = \Upsilon \quad (7.5)$$

with $\Upsilon(\phi, \bar{\phi})$ a Killing potential [49].

Consider now a hyperkähler manifold $(\mathcal{M}, g_{\mu\nu}, J_A^\mu{}_\nu)$, where $J_A^\mu{}_\nu$ are the three integrable complex structures obeying the quaternion algebra (6.50). On such a manifold we can define a tri-holomorphic vector field V that is holomorphic with respect to all the complex structures. Let us choose local complex coordinates, $\phi^\mu = (\phi^a, \bar{\phi}^{\bar{a}})$, such that the complex structure J_3 becomes diagonal, eq. (6.48), and the other complex structures can be brought to the form (6.49). As before, holomorphy with respect to J_3 means that the component V^a of the vector field (7.4) is holomorphic, $V^a = V^a(\phi)$. Holomorphy with respect to J_1 amounts to the conditions

$$0 = \omega^{c\bar{b}} \nabla_c V^a - \omega^a_{\bar{c}} \nabla_{\bar{b}} V^{\bar{c}} \quad (7.6a)$$

$$0 = \omega^{\bar{c}b} \nabla_{\bar{c}} V^a - \omega^a_{\bar{c}} \nabla_b V^{\bar{c}} \quad (7.6b)$$

along with their complex conjugates. Holomorphy with respect to J_2 amounts to

$$0 = \omega_{\bar{b}}^c \nabla_c V^a - \omega_{\bar{c}}^a \nabla_{\bar{b}} V^{\bar{c}} \quad (7.7a)$$

$$0 = \omega_{\bar{b}}^{\bar{c}} \nabla_{\bar{c}} V^a + \omega_{\bar{c}}^a \nabla_b V^{\bar{c}} . \quad (7.7b)$$

Note that (7.6a) and (7.7a) are identical while (7.6b) and (7.7b) differ in the relative sign of the two terms. If V is holomorphic with respect to J_3 , then the conditions (7.6b) and (7.7b) are satisfied. It is easy to check that if a vector is holomorphic with respect to any two of these complex structures, it is automatically holomorphic with respect to the third.

7.2 Superpotential in $\mathcal{N} = 2$ rigid supersymmetric theories

Tri-holomorphic Killing vector fields naturally occur in $\mathcal{N} = 2$ rigid supersymmetric σ -models with non-vanishing superpotentials. Such a model is described by the action

$$S = \int d^4x d^4\theta K(\phi, \bar{\phi}) + \int d^4x d^2\theta W(\phi) + \int d^4x d^2\bar{\theta} \bar{W}(\bar{\phi}) . \quad (7.8)$$

As is well-known (see [17] and references therein), $\mathcal{N} = 2$ supersymmetry requires that (i) $K(\phi, \bar{\phi})$ is the Kähler potential of a hyperkähler manifold \mathcal{M} ; (ii) $W(\phi)$ must be such that

$$V^a = \omega^{ab} W_b , \quad V^{\bar{a}} = \bar{\omega}^{\bar{a}\bar{b}} \bar{W}_{\bar{b}} \quad (7.9)$$

is a tri-holomorphic Killing vector field on \mathcal{M} . Holomorphy with respect to J_3 means that $V^a = V^a(\phi)$, while holomorphy with respect to J_1 (and J_2) amount to

$$\omega_{\bar{a}}^c \nabla_b W_c + \omega_b^{\bar{c}} \nabla_{\bar{a}} \bar{W}_{\bar{c}} = 0 . \quad (7.10)$$

The Killing equations

$$\nabla_a V_b + \nabla_b V_a = \nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0 \quad (7.11)$$

are satisfied as a consequence.

7.3 Geometry of $\mathcal{N} = 2$ AdS σ -models

The geometric structure we have uncovered in AdS is quite interesting. First of all, we have found that AdS supersymmetry demands the existence of a vector field $V^\mu = (V^a, V^{\bar{a}})$ of the form

$$V^a = \frac{\mu}{2|\mu|} \omega^{ab} \mathcal{K}_b , \quad V^{\bar{a}} = \frac{\bar{\mu}}{2|\mu|} \bar{\omega}^{\bar{a}\bar{b}} \mathcal{K}_{\bar{b}} , \quad (7.12)$$

which obeys the Killing equations

$$\nabla_a V_b + \nabla_b V_a = \nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0 . \quad (7.13)$$

It is clearly not holomorphic with respect to J_3 . In fact, it is easy to show that V *rotates* the complex structures:

$$\mathcal{L}_V J_1 = \frac{\text{Im } \mu}{|\mu|} J_3 = J_3 \sin \theta \quad (7.14a)$$

$$\mathcal{L}_V J_2 = -\frac{\text{Re } \mu}{|\mu|} J_3 = -J_3 \cos \theta \quad (7.14b)$$

$$\mathcal{L}_V J_3 = \frac{\text{Re } \mu}{|\mu|} J_2 - \frac{\text{Im } \mu}{|\mu|} J_1 = J_2 \cos \theta - J_1 \sin \theta \quad (7.14c)$$

where $\theta := \arg \mu$. There is a preferred complex structure

$$J_{\text{AdS}} := \frac{\text{Re } \mu}{|\mu|} J_1 + \frac{\text{Im } \mu}{|\mu|} J_2 = \frac{1}{|\mu|} \begin{pmatrix} 0 & \mu \omega^a_{\bar{b}} \\ \bar{\mu} \omega^{\bar{a}}_b & 0 \end{pmatrix} \quad (7.15)$$

(normalized as usual so that $J^2 = -\mathbb{1}$) with respect to which V^μ is *holomorphic*,

$$\mathcal{L}_V J_{\text{AdS}} = 0 . \quad (7.16)$$

It turns out that the conditions (7.14), which imply (7.16), *also* imply (7.12) in an elementary way. As a consequence of (7.16), one can always introduce some function \mathcal{K} so that

$$V^\mu = \frac{1}{2} J_{\text{AdS}}{}^\mu{}_\nu \nabla^\nu \mathcal{K} . \quad (7.17)$$

This function \mathcal{K} is (up to a numerical factor) the real *Killing potential* for V^μ as defined in (7.5), if we work in the basis where J_{AdS} is diagonal. Moreover, using eqs. (7.14), it is a simple exercise to show that

$$g_{\mu\nu} = \frac{1}{2} (\delta_\mu{}^\rho \delta_\nu{}^\sigma + J_{3\mu}{}^\rho J_{3\nu}{}^\sigma) \nabla_\rho \nabla_\sigma \mathcal{K} \quad (7.18)$$

or equivalently (in complex coordinates where J_3 is diagonalized)

$$g_{ab} = 0 , \quad g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K} . \quad (7.19)$$

In other words, the function \mathcal{K} is not only the Killing potential with respect to J_{AdS} , but *also* the *Kähler potential* with respect to J_3 . In fact, it is the Kähler potential with respect to *any* complex structure orthogonal to J_{AdS} . Thus the specification of a Killing vector (7.12) in terms of the Kähler potential \mathcal{K} is *completely equivalent* to the

geometric requirement that the hyperkähler manifold permit a Killing vector which rotates the complex structures as (7.14).¹⁵ In accordance with our earlier discussion, the function \mathcal{K} must be globally defined, and therefore the Kähler two-form associated with *any* complex structure orthogonal to J_{AdS} must be exact.

It is quite remarkable that many of the features described above have also been noticed recently in the context of supersymmetric nonlinear σ -models in AdS_5 [50, 51]. As argued in [50], the AdS_5 supersymmetry requires the σ -model target space to be hyperkähler and possess a holomorphic Killing vector field (i.e. holomorphic with respect to J_3). It was noted in [51] that in fact, the holomorphic Killing vector field acts as a *rotation* on the complex structures.

It is also worth mentioning that there is an interesting formal similarity between (7.9) and (7.12). Recall that the general AdS Lagrangian \mathcal{K} may be interpreted as arising from a real Kähler potential K and a holomorphic superpotential W via

$$\mathcal{K} = K + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}} \quad (7.20)$$

where K and W transform under Kähler transformations as

$$K \rightarrow K + F + \bar{F} \ , \quad W \rightarrow W - \mu F \ . \quad (7.21)$$

Because W transforms nonlinearly, the Kähler-covariant derivative of W is naturally defined as

$$\nabla_a W := \partial_a W + \mu \partial_a K. \quad (7.22)$$

However, one easily sees that

$$\nabla_a W = \mu \partial_a \mathcal{K} \quad (7.23)$$

and so V^a can be equally as well written

$$V^a = \frac{1}{2|\mu|} \omega^{ab} \nabla_b W \ . \quad (7.24)$$

This formally resembles (7.9) (up to the factor of $1/2|\mu|$). In AdS, the obstruction to tri-holomorphy arises since $[\nabla_{\bar{a}}, \nabla_b]$ involves a curvature associated with Kähler transformations. Thus,

$$\nabla_{\bar{a}} \nabla_b W = [\nabla_{\bar{a}}, \nabla_b] W = \mu g_{\bar{a}b} \neq 0 \quad (7.25)$$

implies that V^a cannot be tri-holomorphic. In fact, it acts as a rotation on the complex structures (7.14).

¹⁵Hyperkähler manifolds with such properties were discussed by Hitchin et al. [43].

¹⁵The Killing vector turns out to be holomorphic due to a certain imbedding of the hypermultiplets into 4D $\mathcal{N} = 1$ chiral superfields.

7.4 Retrofitting hyperkähler metrics and nonlinear σ -models

We have identified the key criterion for whether a hyperkähler metric may be used as the target space for the a nonlinear σ -model in AdS: it must possess a Killing vector which rotates the complex structures. Moreover, we have even found the Lagrangian \mathcal{K} : it is the Killing potential with respect to the invariant complex structure. Given a hyperkähler manifold with the requisite geometric property, we may directly construct (at least in principle) the correct Lagrangian \mathcal{K} .

To demonstrate this, we take the simplest example possible: a four-dimensional hyperkähler manifold with vanishing curvature. This space is easily described by the Kähler potential $K = x\bar{x} + y\bar{y}$ for complex coordinates x and y . This manifold is naturally equipped with a canonical holomorphic two-form

$$\omega_{ab} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (7.26)$$

This manifold possesses a $U(2)$ group of holomorphic isometries associated with the Killing vectors

$$V_0^a = (ix, iy) , \quad V_1^a = (iy, ix) , \quad V_2^a = (y, -x), \quad V_3^a = (ix, -iy) . \quad (7.27)$$

The Killing vectors $V_I = \{V_1, V_2, V_3\}$ obey an $SU(2)$ algebra and are tri-holomorphic. The Killing vector V_0 is holomorphic with respect to J_3 alone and acts as a rotation in the plane of J_1 and J_2 ,

$$\mathcal{L}_{V_0} J_1 = -2J_2 , \quad \mathcal{L}_{V_0} J_2 = +2J_1 . \quad (7.28)$$

Now let us introduce a new Killing vector V_{AdS} given by

$$V_{\text{AdS}} = \frac{1}{2}V_0 + \frac{1}{2}c_I V_I \quad (7.29)$$

where c_I are arbitrary real constants. Let us denote $J_{\text{AdS}} = J_3$ in this basis. Because V_I are tri-holomorphic, this Killing vector rotates the complex structures around the axis selected out by J_{AdS} . Since V_{AdS} is a holomorphic Killing vector in this basis, we can easily construct its Killing potential, using

$$V_{\text{AdS}}^\mu = \frac{1}{2}J_{\text{AdS}}^\mu{}_\nu \nabla^\nu \mathcal{K} . \quad (7.30)$$

The result is

$$\mathcal{K} = x\bar{x} + y\bar{y} + c_1(x\bar{y} + y\bar{x}) + ic_2(x\bar{y} - y\bar{x}) + c_3(x\bar{x} - y\bar{y}) . \quad (7.31)$$

In the basis where J_{AdS} is given by (7.15)

$$J_{\text{AdS}} = \frac{1}{|\mu|} \begin{pmatrix} 0 & i\mu\sigma_2 \\ i\bar{\mu}\sigma_2 & 0 \end{pmatrix}, \quad (7.32)$$

the function \mathcal{K} will coincide with an acceptable AdS Lagrangian. It is easy to construct this new coordinate basis. Let ϕ and ψ be new complex coordinates given by

$$x = \frac{1}{\sqrt{2|\mu|}} (\bar{\mu}^{1/2}\phi - i\mu^{1/2}\bar{\psi}), \quad (7.33a)$$

$$y = \frac{1}{\sqrt{2|\mu|}} (\bar{\mu}^{1/2}\psi + i\mu^{1/2}\bar{\phi}). \quad (7.33b)$$

It is easy to see that in the new complex coordinates, the metric remains Kähler and of canonical form. It again possesses a holomorphic two-form given by (7.26). We easily find now that \mathcal{K} is given by

$$\mathcal{K} = \phi\bar{\phi} + \psi\bar{\psi} + \frac{1}{|\mu|} \left(\bar{\mu}\phi^2(c_2 - ic_1) + \bar{\mu}\psi^2(c_2 + ic_1) + ic_3\bar{\mu}\phi\psi + \text{c.c.} \right). \quad (7.34)$$

For the choice $c_1 = c_2 = 0$ and $c_3 = m/|\mu|$, this function \mathcal{K} matches the free off-shell $\mathcal{N} = 2$ hypermultiplet model given in eq. (6.11). For other values of c_I , we have found a family of allowed AdS Lagrangians with a flat hyperkähler metric. It is also hardly a coincidence that the terms proportional to c_I are, in this final formulation, the real part of a holomorphic function. This is simply a consequence of the Killing vectors V_I being tri-holomorphic.

Although this was a fairly simple example, the technique appears to be quite general. As a *nontrivial* example, let us consider the target space hyperkähler metric associated with the Eguchi-Hanson gravitational instanton in four dimensions. It possesses a Kähler potential (see e.g. [15])

$$K = \sqrt{a^4 + \rho^4} - a^2 \log \left(\frac{a^2 + \sqrt{a^4 + \rho^4}}{\rho^2} \right), \quad \rho^2 := x\bar{x} + y\bar{y}, \quad (7.35)$$

in terms of two complex coordinates x and y . The dimensionful parameter a represents the size of the gravitational instanton in the target space. For $\rho \gg a$, the Kähler potential reduces to the free hypermultiplet.

In these coordinates, the Kähler metric and its inverse are given by

$$g_{a\bar{b}} = \frac{1}{\rho^4 \sqrt{a^4 + \rho^4}} \begin{pmatrix} \rho^6 + a^4 y\bar{y} & -a^4 y\bar{x} \\ -a^4 x\bar{y} & \rho^6 + a^4 x\bar{x} \end{pmatrix}, \quad (7.36a)$$

$$g^{\bar{a}b} = \frac{1}{\rho^4 \sqrt{a^4 + \rho^4}} \begin{pmatrix} \rho^6 + a^4 x\bar{x} & a^4 \bar{x}y \\ a^4 \bar{y}x & \rho^6 + a^4 y\bar{y} \end{pmatrix}. \quad (7.36b)$$

The holomorphic two-form ω_{ab} again has the canonical form

$$\omega_{ab} = \omega_{\bar{a}\bar{b}} = \omega^{ab} = \omega^{\bar{a}\bar{b}} = i\sigma_2 . \quad (7.37)$$

As before, there is a family of holomorphic isometries involving the vectors V_0 and V_I . V_I are tri-holomorphic while V_0 rotates the complex structures. Taking the same combination as before for V_{AdS} , we are led to construct the Killing potential \mathcal{K} as

$$\mathcal{K} = \frac{\sqrt{a^4 + \rho^4}}{\rho^2} \left(\rho^2 + c_1(x\bar{y} + y\bar{x}) + ic_2(x\bar{y} - y\bar{x}) + c_3(x\bar{x} - y\bar{y}) \right) . \quad (7.38)$$

Evidently this scalar function, when rewritten in the new coordinates which transform J_{AdS} to (7.32), is an AdS Lagrangian with an Eguchi-Hanson target space; furthermore, the terms proportional to c_I are the real part of a holomorphic function in the new complex coordinates. What makes this procedure nontrivial is *finding* these complex coordinates. One set was given, e.g., in [52].

An important feature worthy of note is the behavior of these two Kähler potentials as ρ tends to zero. The original potential (7.35) is clearly singular at $\rho = 0$. The coordinates x and y cover only a single patch of the full target space manifold and a Kähler transformation is necessary to connect the different patches. However, the function \mathcal{K} , eq. (7.38), in order to be a physical AdS Lagrangian, must be globally defined. Indeed we see that it possesses a well-defined $\rho \rightarrow 0$ limit.

7.5 Hyperkähler structure of dual tensor models

We are now prepared to resume the study of the dual tensor model in AdS derived in subsection 4.4. The original tensor model Lagrangian $L(\varphi, \bar{\varphi}, G)$ obeyed the conditions

$$\frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial G^I \partial G^J} = 0 , \quad (7.39)$$

$$\frac{\partial^2 L}{\partial \varphi^I \partial G^J} - \frac{\partial^2 L}{\partial \varphi^J \partial G^I} = F_{IJ}(\varphi) , \quad (7.40)$$

where F_{IJ} is an exact holomorphic two-form. As discussed already, the second condition follows from the first and one can always make a trivial transformation on the Lagrangian to set $F_{IJ} = 0$. However, we will keep a nonzero value for full generality.

It is a straightforward task to calculate the Kähler metric in the dual geometry.

The result is [43]

$$\frac{\partial^2 \mathcal{K}}{\partial \varphi^I \partial \bar{\varphi}^J} = \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial \varphi^I \partial G^K} \left(\frac{\partial^2 L}{\partial \varphi^K \partial \bar{\varphi}^J} \right)^{-1} \frac{\partial^2 L}{\partial G^K \partial \bar{\varphi}^J} , \quad (7.41a)$$

$$\frac{\partial^2 \mathcal{K}}{\partial \varphi^I \partial \bar{\psi}_J} = - \frac{\partial^2 L}{\partial \varphi^I \partial G^K} \left(\frac{\partial^2 L}{\partial \varphi^K \partial \bar{\varphi}^J} \right)^{-1} , \quad (7.41b)$$

$$\frac{\partial^2 \mathcal{K}}{\partial \psi_I \partial \bar{\varphi}^J} = - \left(\frac{\partial^2 L}{\partial \varphi^K \partial \bar{\varphi}^I} \right)^{-1} \frac{\partial^2 L}{\partial G^K \partial \bar{\varphi}^J} , \quad (7.41c)$$

$$\frac{\partial^2 \mathcal{K}}{\partial \psi_I \partial \bar{\psi}_J} = \left(\frac{\partial^2 L}{\partial \bar{\varphi}^J \partial \varphi^I} \right)^{-1} . \quad (7.41d)$$

We may cast this in matrix form by introducing

$$(\mathbf{M})_{IJ} := \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} = - \frac{\partial^2 L}{\partial G^I \partial G^J} , \quad (7.42a)$$

$$(\mathbf{Q})_{IJ} := \frac{\partial^2 L}{\partial \varphi^I \partial G^J} , \quad (7.42b)$$

$$(\mathbf{Q}^\dagger)_{IJ} = \frac{\partial^2 L}{\partial G^I \partial \bar{\varphi}^J} . \quad (7.42c)$$

Note that \mathbf{M} is symmetric and real. The Kähler metric may be written in matrix form as

$$g_{a\bar{b}} := \begin{pmatrix} \mathbf{M} + \mathbf{Q}\mathbf{M}^{-1}\mathbf{Q}^\dagger & -\mathbf{Q}\mathbf{M}^{-1} \\ -\mathbf{M}^{-1}\mathbf{Q}^\dagger & \mathbf{M}^{-1} \end{pmatrix} . \quad (7.43)$$

In this matrix form, the Monge-Ampère equation $\det g_{a\bar{b}} = 1$, see e.g. [43, 48], is clearly obeyed. The inverse Kähler metric also takes a very simple form

$$g^{\bar{a}b} = \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{M}^{-1}\mathbf{Q} \\ \mathbf{Q}^\dagger\mathbf{M}^{-1} & \mathbf{M} + \mathbf{Q}^\dagger\mathbf{M}^{-1}\mathbf{Q} \end{pmatrix} . \quad (7.44)$$

The complex structure in the dual model is easily calculated. In similar notation, it is given by

$$\omega^{ab} = \begin{pmatrix} \mathbf{0} & -\mathbb{1} \\ \mathbb{1} & \mathbf{F} \end{pmatrix} , \quad \omega_{ab} = \begin{pmatrix} -\mathbf{F} & -\mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix} \quad (7.45)$$

where $(\mathbf{F})_{IJ} := F_{IJ}(\varphi)$. It is straightforward to check that the complex structure is covariantly constant. Because F_{IJ} may always be taken to zero by shifting the original Lagrangian by a certain H -transformation, one can always choose the dual complex structures to be of a simple canonical form.

We already know that any $\mathcal{N} = 2$ nonlinear σ -model in AdS must possess a special Killing vector V^μ which acts as a rotation on the complex structures. It is enlightening to see how this comes about for nonlinear σ -models which are dual to $\mathcal{N} = 2$ tensor multiplet models. We easily calculate $V^\mu = (V^a, V^{\bar{a}})$ where

$$V^a = \begin{cases} -\frac{\mu}{2|\mu|} \frac{\partial \mathcal{K}}{\partial \psi_I} , & a = I \\ \frac{\mu}{2|\mu|} \left(\frac{\partial \mathcal{K}}{\partial \varphi^I} + F_{IJ} \frac{\partial \mathcal{K}}{\partial \psi_J} \right) , & a = I + n . \end{cases} \quad (7.46)$$

Using the solution $G^I = G^I(\varphi, \bar{\varphi}, \psi + \bar{\psi})$ from the duality transformation, the first term above can be rewritten as

$$V^a = \frac{\mu}{2|\mu|} G^I , \quad a = I \quad (7.47)$$

and so the Killing vector acts on the coordinates φ as $V\varphi^I = \mu G^I / 2|\mu|$. If we calculate how the Killing vector acts on the function $G^I(\varphi, \bar{\varphi}, \psi + \bar{\psi})$, we find

$$V^\mu \partial_\mu G^I = -\frac{1}{|\mu|} (\bar{\mu} \varphi^I + \mu \bar{\varphi}^I) . \quad (7.48)$$

If we interpret this as a transformation of the original tensor variables $(\varphi^I, \bar{\varphi}^I, G^I)$, this is simply an $\text{SO}(2)$ transformation rotating the $\mathcal{N} = 2$ tensor multiplet! Unsurprisingly, the specific $\text{SO}(2)$ subgroup of $\text{SU}(2)$ appearing here is that which preserves the form of \mathbf{S}^{ij} in the underlying projective superspace description of section 5.

Using the complex structure and Kähler metric, we may calculate $V_\mu = (V_a, V_{\bar{a}})$. One finds

$$2|\mu|V_{\bar{a}} = \begin{cases} \mu \frac{\partial^2 \mathcal{L}}{\partial \varphi^I \partial \bar{\varphi}^J} G^J + \mu \frac{\partial^2 \mathcal{L}}{\partial \bar{\varphi}^I \partial G^J} \left(\left(\frac{\partial^2 L}{\partial \varphi^J \partial \bar{\varphi}^I} \right)^{-1} C_J + 2\bar{\varphi}^I \right) , & a = I \\ \mu \left(\frac{\partial^2 L}{\partial \varphi^J \partial \bar{\varphi}^I} \right)^{-1} C_J + 2\mu \bar{\varphi}^I , & a = I + n \end{cases} \quad (7.49)$$

where

$$C_J := \frac{\partial L}{\partial \varphi^I} - \frac{\partial^2 L}{\partial \varphi^I \partial G^J} G^J - 2 \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} \bar{\varphi}^J . \quad (7.50)$$

Recall that the additional AdS condition for tensor models (4.21) amounts to

$$R_J = \mu C_J + \bar{\mu} \bar{C}_J = 0 \quad (7.51)$$

and so, for example,

$$V_a + V_{\bar{a}} = \frac{1}{|\mu|} (\mu \bar{\varphi}^I + \bar{\mu} \varphi^I) , \quad a = I + n . \quad (7.52)$$

In order for V^μ to be a Killing vector, it must obey a number of conditions. The only nontrivial one is $\partial_a V_{\bar{b}} + \partial_{\bar{b}} V_a = 0$. This is straightforward (but tedious) to check.

8 $\mathcal{N} = 2$ superconformal σ -models

Both Minkowski and AdS $\mathcal{N} = 2$ superspaces have the same superconformal group $SU(2, 2|2)$. Thus all $\mathcal{N} = 2$ rigid superconformal σ -models should be invariant under the $\mathcal{N} = 2$ AdS supergroup $OSp(2|4)$. Here we elaborate on this point.

8.1 Hyperkähler cones

Target spaces for $\mathcal{N} = 2$ superconformal σ -models are hyperkähler cones (see [8, 9, 10] and references therein). By definition, a hyperkähler cone is a hyperkähler manifold possessing an infinitesimal dilatation or, equivalently, a homothetic conformal Killing vector field which is the gradient of a function. Let us recall the salient facts about homothetic conformal Killing vector fields (see [7, 10] for more details).

Consider a Riemannian manifold $(\mathcal{M}, g_{\mu\nu})$ admitting an infinitesimal dilatation. The required homothetic conformal Killing vector field $\chi = \chi^\mu \partial_\mu$ is defined to obey the equation

$$\nabla_\nu \chi^\mu = \delta_\nu^\mu \iff \nabla_\nu \chi_\mu = g_{\nu\mu} , \quad (8.1)$$

and hence

$$\chi_\mu = \frac{1}{2} \nabla_\mu \chi^2 , \quad \chi^2 = g_{\mu\nu} \chi^\mu \chi^\nu . \quad (8.2)$$

The manifold \mathcal{M} is globally a (Riemannian) cone [7].

We next specify the Riemannian manifold under consideration to be a Kähler space, $(\mathcal{M}, g_{\mu\nu}, J^\mu{}_\nu)$, with $J^\mu{}_\nu$ the complex structure. We choose local complex coordinates, $\phi^\mu = (\phi^a, \bar{\phi}^{\bar{a}})$, such that the complex structure becomes diagonal, and the Kähler metric takes the form $ds^2 = 2g_{a\bar{b}} d\phi^a d\bar{\phi}^{\bar{b}}$. The required homothetic conformal Killing vector field

$$\chi = \chi^\mu \frac{\partial}{\partial \phi^\mu} = \chi^a \frac{\partial}{\partial \phi^a} + \bar{\chi}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \quad (8.3)$$

is defined to obey the constraint

$$\nabla_\nu \chi^\mu = \delta_\nu^\mu \iff \nabla_b \chi^a = \delta_b^a , \quad \nabla_{\bar{b}} \chi^a = \partial_{\bar{b}} \chi^a = 0 . \quad (8.4)$$

In particular, the vector field χ is holomorphic, $\chi^a = \chi^a(\phi)$. Its properties include:

$$\chi_\mu := g_{\mu\nu} \chi^\nu = \partial_\mu \mathcal{K} \iff \chi_a := g_{a\bar{b}} \bar{\chi}^{\bar{b}} = \partial_a \mathcal{K} , \quad (8.5a)$$

$$\frac{1}{2} g_{\mu\nu} \chi^\mu \chi^\nu = \mathcal{K} \iff g_{a\bar{b}} \chi^a \bar{\chi}^{\bar{b}} = \mathcal{K} . \quad (8.5b)$$

It follows from the above properties that $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$, and thus \mathcal{K} is the Kähler potential. These relations imply that

$$\chi^a(\phi) \partial_a \mathcal{K}(\phi, \bar{\phi}) = \mathcal{K}(\phi, \bar{\phi}) . \quad (8.6)$$

The Kähler potential \mathcal{K} is a globally defined scalar function over \mathcal{M} , in accordance with (8.5b).¹⁶ This implies that the Kähler two-form, $\Omega = 2i g_{a\bar{b}} d\phi^a \wedge d\bar{\phi}^{\bar{b}}$, associated with the Kähler metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$ is exact, and hence \mathcal{M} is necessarily non-compact. This manifold is called a Kählerian cone [7]. It should be remarked that associated with the conformal Killing vector χ is a U(1) Killing vector

$$X = i \chi^a(\phi) \frac{\partial}{\partial \phi^a} - i \bar{\chi}^{\bar{a}}(\bar{\phi}) \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} , \quad (8.7)$$

which leaves the Kähler potential invariant,

$$X\mathcal{K} = 0 . \quad (8.8)$$

A hyperkähler cone is simply a hyperkähler manifold $(\mathcal{M}, g_{\mu\nu}, J_A^\mu{}_\nu)$ admitting an infinitesimal dilatation. Here $J_A^\mu{}_\nu$ are the three integrable complex structures obeying the quaternion algebra (6.50). Associated with the conformal Killing vector field χ are three Killing vectors $X_A^\mu := J_A^\mu{}_\nu \chi^\nu$, which leave the Kähler potential invariant, $X_A^\mu \partial_\mu \mathcal{K} = 0$. These obey the SU(2) algebra

$$[X_A, X_B] = -2\epsilon_{ABC} X_C . \quad (8.9)$$

8.2 The AdS condition

Given a hyperkähler cone \mathcal{M} , our goal is to make use of the above properties of χ to show that

$$V^\mu = (V^a, V^{\bar{a}}) := \left(\frac{\mu}{2|\mu|} \omega^{ab} \mathcal{K}_b, \frac{\bar{\mu}}{2|\mu|} \omega^{\bar{a}\bar{b}} \mathcal{K}_{\bar{b}} \right) = \left(\frac{\mu}{2|\mu|} \omega^{ab} \chi_b, \frac{\bar{\mu}}{2|\mu|} \omega^{\bar{a}\bar{b}} \chi_{\bar{b}} \right) \quad (8.10)$$

is a Killing vector field, for any non-zero complex parameter μ . By representing $V_a = \bar{\mu} \omega_{ab} \chi^b / 2|\mu|$ and using the facts that ω_{ab} and χ^b are holomorphic, the conditions (7.13) follow. It is also straightforward to check (7.14).

It is instructive to give a slightly different proof that (8.10) is a Killing vector and which shows that V belongs to the Lie algebra of the group SU(2) isometrically

¹⁶Although we call \mathcal{K} the Kähler potential, it should be kept in mind that there is no Kähler invariance, for \mathcal{K} is uniquely determined, eq. (8.5b).

acting on the hyperkähler cone. As shown e.g. in [10, 7], associated with the complex structures $(J_A)^\mu{}_\nu$, eqs. (6.48) and (6.49), are the three Killing vectors $X_A^\mu := (J_A)^\mu{}_\nu \chi^\nu$ which span the Lie algebra of $SU(2)$. In particular, we have that $X_1^\mu = (\omega^{ab}\mathcal{K}_b, \omega^{\bar{a}\bar{b}}\mathcal{K}_{\bar{b}})$ and $X_2^\mu = (i\omega^{ab}\mathcal{K}_b, -i\omega^{\bar{a}\bar{b}}\mathcal{K}_{\bar{b}})$ are Killing vectors. Moreover, it is a simple exercise to check that

$$\mathcal{L}_{X_A} J_B = -[J_A, J_B] = -2\epsilon_{ABC} J_C . \quad (8.11)$$

In the superconformal case there is a unique scalar function \mathcal{K} which serves as the *Killing potential* for each $SU(2)$ isometry and the *Kähler potential* for each complex structure.

The Killing vector (8.10) particular to the AdS case is simply a real combination of $X_1 = (J_1)^\mu{}_\nu \chi^\nu$ and $X_2 = (J_2)^\mu{}_\nu \chi^\nu$, and thus V^μ belongs to the Lie algebra of $SU(2)$ and acts as a rotation on the complex structures.

8.3 Superconformal invariance

Let $\mathcal{K}(\phi^a, \bar{\phi}^{\bar{b}})$ be the Kähler potential of a hyperkähler cone. We demonstrate here that the σ -model

$$S = \int d^4x d^4\theta E \mathcal{K}(\phi, \bar{\phi}) \quad (8.12)$$

is $\mathcal{N} = 2$ superconformal. As shown in Appendix B, an $\mathcal{N} = 2$ superconformal transformation is described in $\mathcal{N} = 2$ AdS superspace in terms of an $\mathcal{N} = 2$ superconformal Killing vector. Upon reduction to $\mathcal{N} = 1$ AdS superspace, such a transformation turns into three different ones: (i) an $\mathcal{N} = 1$ superconformal transformation; (ii) an extended superconformal transformation; and (iii) a shadow chiral rotation. See subsection B.2 for more details.

The action (8.12) is invariant under the $\mathcal{N} = 1$ superconformal transformation

$$\delta_\xi \phi^a = -\xi \phi^a - \sigma \chi^a(\phi) , \quad (8.13)$$

with $\xi = \xi^a \mathcal{D}_a + \xi^\alpha \mathcal{D}_\alpha + \bar{\xi}_{\bar{\alpha}} \bar{\mathcal{D}}^{\bar{\alpha}}$ an arbitrary $\mathcal{N} = 1$ superconformal Killing vector, and the covariantly chiral parameter σ defined by (A.12). This invariance follows from the homogeneity condition (8.6). The action (8.12) is also invariant under the shadow chiral rotation of ϕ^a :

$$\delta \phi^a = \frac{i\alpha}{2} \chi^a(\phi) , \quad \bar{\alpha} = \alpha , \quad (8.14)$$

as a consequence of the identity (8.6). It should be remarked that the shadow chiral rotation is generated by the Killing vector (8.7).

Finally, we define the extended superconformal transformation of the chiral fields:

$$\delta_{\rho, \bar{\rho}} \phi^a = \frac{1}{2} (\bar{\mathcal{D}}^2 - 4\mu) (\bar{\rho} \omega^{ab} \chi_b) \quad (8.15)$$

where $\bar{\rho}$ is an $\mathcal{N} = 1$ superfield obeying certain constraints (see eq. (B.30) for its definition and (B.31) and (B.32) for its constraints). In the flat superspace limit, this correctly reduces to the transformation given in [18]. The invariance of (8.12) under this transformation can be proved in complete analogy with the rigid supersymmetric case [18]. It is sufficient to evaluate the variation $\delta_{\bar{\rho}} S$ which corresponds to the choice $\rho = 0$ and $\bar{\rho} \neq 0$, since the parameters ρ and $\bar{\rho}$ are independent. The variation of the action is

$$\begin{aligned} \delta_{\bar{\rho}} S &= -\frac{1}{2} \int d^4x d^4\theta E (\bar{\mathcal{D}}_{\dot{\alpha}} \chi_a) (\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\rho}) \omega^{ab} \chi_b = -\frac{1}{2} \int d^4x d^4\theta E \bar{\rho}_{\dot{\alpha}} (\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{c}}) g_{\bar{c}a} \omega^{ab} \chi_b \\ &= -\frac{1}{2} \int d^4x d^4\theta E \bar{\rho}_{\dot{\alpha}} (\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}}) \bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}} . \end{aligned} \quad (8.16)$$

Since the tensor fields $\bar{\omega}_{\bar{a}\bar{b}}$ and $\bar{\chi}^{\bar{b}}$ are anti-holomorphic, and the parameter $\bar{\rho}_{\dot{\alpha}}$ is antichiral, the combination $\bar{\rho}_{\dot{\alpha}} \bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}}$ appearing in the integrand is antichiral. Therefore, antichirally projecting the variation gives

$$\delta_{\bar{\rho}} S = \frac{1}{8} \int d^4x d^2\bar{\theta} \bar{\mathcal{E}} \bar{\rho}_{\dot{\alpha}} \bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}} (\mathcal{D}^2 - 4\bar{\mu}) \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} = 0 , \quad (8.17)$$

for $(\mathcal{D}^2 - 4\bar{\mu}) \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Phi}$ is identically zero for any covariantly antichiral superfield $\bar{\Phi}$.

In the case of $\mathcal{N} = 1$ supersymmetry, more general superconformal σ -models exist than those described by the action (8.12) in which the Lagrangian is subject to the homogeneity condition (8.6). In fact, the most general $\mathcal{N} = 1$ superconformal σ -model is given by

$$S = \int d^4x d^4\theta E \mathbb{K}(\phi, \bar{\phi}) , \quad (8.18)$$

where the Lagrangian obeys a generalized homogeneity condition

$$\chi^a(\phi) \partial_a \mathbb{K}(\phi, \bar{\phi}) = \mathbb{K}(\phi, \bar{\phi}) + \frac{2}{\mu} \mathcal{W}(\phi) - \frac{1}{\bar{\mu}} \bar{\mathcal{W}}(\bar{\phi}) , \quad (8.19a)$$

for some homogeneous holomorphic function $\mathcal{W}(\phi)$ of degree three,

$$\chi^a(\phi) \partial_a \mathcal{W}(\phi) = 3\mathcal{W}(\phi) . \quad (8.19b)$$

The general solution of eq. (8.19a) is

$$\mathbb{K}(\phi, \bar{\phi}) = \mathcal{K}(\phi, \bar{\phi}) + \frac{1}{\mu} \mathcal{W}(\phi) + \frac{1}{\bar{\mu}} \bar{\mathcal{W}}(\bar{\phi}) , \quad (8.20)$$

with $\mathcal{K}(\phi, \bar{\phi})$ obeying the homogeneity condition (8.6). The above model in AdS can easily be related to the most general $\mathcal{N} = 1$ superconformal σ -model in Minkowski space

$$S = \int d^4x d^4\theta \mathcal{K}(\phi, \bar{\phi}) + \int d^4x d^2\theta \mathcal{W}(\phi) + \int d^4x d^2\bar{\theta} \bar{\mathcal{W}}(\bar{\phi}) . \quad (8.21)$$

In the case of $\mathcal{N} = 2$ superconformal symmetry, the σ -model action must also be invariant under the shadow chiral rotation (8.14) and the extended superconformal transformation (8.15). These symmetries prove to require the superpotential to vanish,

$$\mathcal{W}(\phi) = 0 . \quad (8.22)$$

8.4 Analysis of the commutation relations

Let us calculate the commutator of two extended superconformal transformations. We find

$$[\delta_2, \delta_1]\phi^a = \frac{1}{2} \xi^{\dot{\alpha}\alpha} \mathcal{D}_{\alpha\dot{\alpha}} \phi^a - \xi^\alpha \mathcal{D}_\alpha \phi^a - \frac{1}{4} (\bar{\mathcal{D}}^2 - 4\mu) \left((\bar{\rho}_1 \rho_2 - \bar{\rho}_2 \rho_1) g^{a\bar{b}} \mathcal{D}^2 \chi_{\bar{b}} \right) \quad (8.23)$$

where

$$\xi_{\alpha\dot{\alpha}} := -4i(\mathcal{D}_\alpha \rho_2 \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\rho}_1 - \mathcal{D}_\alpha \rho_1 \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\rho}_2) , \quad (8.24a)$$

$$\xi_\alpha := \frac{i}{8} \bar{\mathcal{D}}^{\dot{\alpha}} \xi_{\alpha\dot{\alpha}} = 2\mu(\bar{\rho}_1 \mathcal{D}_\alpha \rho_2 - \bar{\rho}_2 \mathcal{D}_\alpha \rho_1) . \quad (8.24b)$$

The third term in (8.23) can be rearranged into a piece which involves the equation of motion and a remainder. The result is

$$\begin{aligned} [\delta_2, \delta_1]\phi^a &= \frac{1}{2} \xi^{\dot{\alpha}\alpha} \mathcal{D}_{\alpha\dot{\alpha}} \phi^a - \xi^\alpha \mathcal{D}_\alpha \phi^a - \sigma \chi^a + \frac{i}{2} \alpha \chi^a \\ &\quad - \frac{1}{4} (\bar{\mathcal{D}}^2 - 4\mu) \left((\bar{\rho}_1 \rho_2 - \bar{\rho}_2 \rho_1) g^{a\bar{b}} (\mathcal{D}^2 - 4\bar{\mu}) \chi_{\bar{b}} \right) \end{aligned} \quad (8.25)$$

where σ is given in terms of $\xi_{\alpha\dot{\alpha}}$ as in (A.12) and α is given by

$$\alpha = 2i(\sigma - \bar{\sigma}) - 8i\mu\bar{\mu}(\bar{\rho}_1 \rho_2 - \bar{\rho}_2 \rho_1) . \quad (8.26)$$

The first line in (8.25) is clearly an $\mathcal{N} = 1$ superconformal transformation combined with a shadow chiral rotation with real parameter α .¹⁷ The second line vanishes on-shell and ensures on-shell closure of the algebra. We see the algebra is open in the superconformal case but becomes closed under the $\text{OSp}(2|4)$ transformations for which $\bar{\rho} = \rho$.

9 $\mathcal{N} = 2$ superfield formulation

One important feature of the $\mathcal{N} = 1$ AdS construction we have presented is that the algebra closes off-shell. For this reason, there ought to exist a formulation in terms of $\mathcal{N} = 2$ superfields. In this section, we present just such a formulation. As a brief warm-up, we describe an AdS generalization of the Fayet-Sohnius hypermultiplet [53, 54].

9.1 Warm-up: Fayet-Sohnius hypermultiplet in AdS_4

Recall the algebra of covariant derivatives in AdS:

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = 4\mathbf{S}^{ij}M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathbf{S}^{kl}J_{kl}, \quad \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}j}\} = -2i\delta_j^i\mathcal{D}_{\alpha\dot{\alpha}}, \quad (9.1a)$$

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_\beta^i] = -i\varepsilon_{\alpha\beta}\mathbf{S}^{ij}\bar{\mathcal{D}}_{\dot{\alpha}j}, \quad [\mathcal{D}_a, \mathcal{D}_b] = -\mathbf{S}^2M_{ab}. \quad (9.1b)$$

A key feature of this algebra is that only an $\text{SO}(2)_R \cong \text{U}(1)_R$ subgroup of $\text{SU}(2)_R$ generated by $J := \mathbf{S}^{kl}J_{kl}$ is respected. The generator J acts as

$$[J, \mathcal{D}_\alpha^i] = \mathbf{S}^i_j \mathcal{D}_\alpha^j, \quad [J, \bar{\mathcal{D}}_i^{\dot{\alpha}}] = -\mathbf{S}_i^j \bar{\mathcal{D}}_j^{\dot{\alpha}}, \quad [J, \mathbf{S}^{ij}] = 0. \quad (9.2)$$

The constant isotriplet \mathbf{S}^{ij} is chosen to obey (B.18) and (B.19).

We introduce the Fayet-Sohnius hypermultiplet q^i , which is defined to obey the constraints

$$\mathcal{D}_\alpha^{(i} q^{j)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(i} q^{j)} = 0. \quad (9.3)$$

However, the action of the generator J on q^i is *not* fixed in advance. It must be determined by the constraints (9.3). Making use of the algebra of covariant derivatives, one can show that

$$J = \frac{1}{4}\{\bar{\mathcal{D}}_{\dot{\alpha}\perp}, \bar{\mathcal{D}}_{\dot{\alpha}\perp}^{\dot{\alpha}}\} \quad (9.4)$$

¹⁷The parameter α must be constant, and this can be shown to be the case using the formulae given in Appendix B.

and so we may easily deduce

$$Jq_{\underline{1}} = -\frac{1}{4}(\bar{\mathcal{D}}_{\underline{1}})^2 q_{\underline{2}} , \quad J\bar{q}_{\underline{1}} = -\frac{1}{4}(\bar{\mathcal{D}}_{\underline{1}})^2 \bar{q}_{\underline{2}} . \quad (9.5)$$

Similarly,

$$J = \frac{1}{4}\{\mathcal{D}^{\alpha\underline{1}}, \mathcal{D}_{\alpha}^{\underline{2}}\} \quad (9.6)$$

which leads to

$$Jq_{\underline{2}} = \frac{1}{4}(\mathcal{D}^{\underline{1}})^2 q_{\underline{1}} , \quad J\bar{q}_{\underline{2}} = \frac{1}{4}(\mathcal{D}^{\underline{1}})^2 \bar{q}_{\underline{1}} . \quad (9.7)$$

We note that the superfield $q_{\underline{1}}$ is $\mathcal{N} = 1$ chiral

$$\bar{\mathcal{D}}_{\dot{\alpha}\underline{1}} q_{\underline{1}} = 0 . \quad (9.8)$$

However, the right-hand side of $Jq_{\underline{1}}$ in (9.5) is *not* chiral. We may rewrite it then in the form

$$Jq_{\underline{1}} + \mu q_{\underline{2}} = -\frac{1}{4}\left[(\bar{\mathcal{D}}_{\underline{1}})^2 - 4\mu\right]q_{\underline{2}} . \quad (9.9)$$

Let us denote by \mathbb{J} the U(1) operator transforming q_i as an isospinor,

$$\mathbb{J} q_i = -\mathbf{S}_i^j q_j = -\mathbf{S}^j_{ij} q_j . \quad (9.10)$$

Then

$$\mathbb{J} q_{\underline{1}} = -\mathbf{S}_{\underline{1}}^{\underline{2}} q_{\underline{2}} = \mathbf{S}^{\underline{2}\underline{2}} q_{\underline{2}} = -\mu q_{\underline{2}} . \quad (9.11)$$

This allows (9.9) to be rewritten as

$$\Delta q_{\underline{1}} = -\frac{1}{4}\left[(\bar{\mathcal{D}}_{\underline{1}})^2 - 4\mu\right]q_{\underline{2}} , \quad \Delta := J - \mathbb{J} . \quad (9.12)$$

The operator Δ commutes with the covariant derivatives, as (9.2) gives

$$[\Delta, \mathcal{D}_{\alpha}^i] = [\Delta, \bar{\mathcal{D}}_{\dot{\alpha}i}] = 0 , \quad (9.13)$$

and therefore it can be interpreted as an intrinsic central charge.

Note also that the first expression in (9.7) can be rewritten

$$\Delta q_{\underline{2}} = \frac{1}{4}\left[(\mathcal{D}^{\underline{1}})^2 - 4\bar{\mu}\right]q_{\underline{1}} . \quad (9.14)$$

Combined with (9.11) and (9.13), this yields

$$(\Delta^2 + \square_c)q_{\underline{1}} = 0 \quad (9.15)$$

where

$$\square_c := \frac{1}{16} \left[(\bar{\mathcal{D}}_{\underline{1}})^2 - 4\mu \right] \left[(\mathcal{D}^{\underline{1}})^2 - 4\bar{\mu} \right] \quad (9.16)$$

is the covariantly chiral d'Alembertian. In the case of massless Fayet-Sohnius hypermultiplet, the equation of motion is $\Delta = 0$. In the massive case, it takes the form $\Delta = im = \text{const}$, with m a real mass parameter.

There are several very important lessons we can take away from this discussion. First we impose the constraints (9.3) and then *derive* the action of the $\text{SO}(2)$ generator on the hypermultiplet as a consequence. However, it is still possible to separate the $\text{SO}(2)$ generator into a “natural” generator, whose action is specified, along with a separate piece Δ which commutes with everything else,

$$\mathbf{S}^{jk} J_{jk} = \mathbb{J} + \Delta . \quad (9.17)$$

In addition, the operator Δ , at least in this example, is constant on-shell.

9.2 $\mathcal{N} = 2$ hypermultiplets as deformed Fayet-Sohnius

We turn now to our real task: constructing an $\mathcal{N} = 2$ superfield formulation for the off-shell structure we have constructed. Recall that we have an $\mathcal{N} = 1$ superfield ϕ^a transforming as

$$\delta\phi^a = \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\bar{\Omega}^a) = \omega^a_{\bar{b}\bar{\varepsilon}\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{b}} + \frac{1}{2}\varepsilon\bar{\mathcal{D}}_{\dot{\alpha}}\left(\omega^a_{\bar{b}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{b}}\right) \quad (9.18)$$

under the second supersymmetry and $\text{SO}(2)$ transformation. On-shell, this takes the simpler form

$$\delta\phi^a = \omega^a_{\bar{b}\bar{\varepsilon}\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{b}} + 4|\mu|\varepsilon V^a \quad (9.19)$$

where $V^a = \mu\omega^{ab}\mathcal{K}_b/2|\mu|$ is a Killing vector.

We would like to interpret ϕ^a as the $\mathcal{N} = 1$ projection of some $\mathcal{N} = 2$ superfield Φ^a . In doing so, we should identify $\delta\phi^a$ as the lowest component of a corresponding $\mathcal{N} = 2$ transformation $\delta\Phi^a$

$$\delta\Phi^a = -\xi_{\underline{2}}^{\alpha}\mathcal{D}_{\alpha}^2\Phi^a - \bar{\xi}_{\dot{\alpha}}^{\underline{2}}\bar{\mathcal{D}}_{\dot{\alpha}}^2\Phi^a - 2\varepsilon\mathbf{S}^{jk}J_{jk}\Phi^a . \quad (9.20)$$

Because of the chirality of ϕ^a and the absence of $\varepsilon^{\alpha} = \xi_{\underline{2}}^{\alpha}$ in (9.18), we are led to conclude

$$\bar{\mathcal{D}}_{\dot{\alpha}\underline{1}}\Phi^a = \mathcal{D}_{\alpha}^2\Phi^a = 0 . \quad (9.21)$$

Similarly, because of the form of the $\bar{\varepsilon}_{\dot{\alpha}} = \xi_{\dot{\alpha}}^2$ term, we must also choose

$$\bar{\mathcal{D}}_{\dot{\alpha}2}\Phi^a = -\omega^a_{\bar{b}}\bar{\mathcal{D}}_{\dot{\alpha}1}\bar{\Phi}^{\bar{b}}. \quad (9.22)$$

It is now a simple task to use these constraints to work out the action of the SO(2) generator. We find

$$\mathbf{S}^{jk}J_{jk}\Phi^a = -\frac{1}{4}\bar{\mathcal{D}}_{\dot{\alpha}1}\left(\omega^a_{\bar{b}}\bar{\mathcal{D}}_{\dot{\alpha}1}\bar{\Phi}^{\bar{b}}\right) = -\frac{1}{4}(\bar{\mathcal{D}}_{\dot{\alpha}1})^2(\omega^{ab}\mathcal{K}_b). \quad (9.23)$$

As with the Fayet-Sohnius case, it is useful to split the SO(2) generator into two pieces, one of which acts upon Φ^a in a geometric way and a remainder which preserves $\mathcal{N} = 1$ chirality and commutes with the covariant derivative. Taking as before $\mathbf{S}^{jk}J_{jk} = \mathbb{J} + \Delta$ we *define*

$$\mathbb{J}\Phi^a := -\mu\omega^{ab}\mathcal{K}_b = -2|\mu|V^a \quad (9.24)$$

in analogy with (9.10). Moreover, one can easily check that the function \mathcal{K} is invariant

$$\mathbb{J}\mathcal{K} = -\mu\omega^{ab}\mathcal{K}_a\mathcal{K}_b + \text{c.c.} = 0 \quad (9.25)$$

under the action of \mathbb{J} . This is exactly as one would expect from an SO(2) generator. From the definition (9.24) it follows that the residual piece Δ given by

$$\Delta\Phi^a = -\frac{1}{4}\left[(\bar{\mathcal{D}}_{\dot{\alpha}1})^2 - 4\mu\right](\omega^{ab}\mathcal{K}_b) \quad (9.26)$$

is chiral, which is consistent with the requirement (9.13). Note also that this quantity vanishes on-shell,

$$\Delta\Phi^a = -\frac{1}{4}\omega^{ab}\left[(\bar{\mathcal{D}}_{\dot{\alpha}1})^2 - 4\mu\right]\mathcal{K}_b = 0 \quad (\text{on-shell}), \quad (9.27)$$

due to the chirality of ω^{ab} .¹⁸

We may provide some additional justification for the choice (9.24) by considering a superconformal model, where this choice is quite natural. For the constraints (9.21) and (9.22) to be consistent with the superconformal algebra, the dilatation generator \mathbb{D} , chiral U(1)_R generator \mathbf{J} , and SU(2)_R generators J_{ij} must act on Φ^a as

$$\mathbb{D}\Phi^a = \chi^a, \quad \mathbf{J}\Phi^a = 0 \quad (9.28a)$$

$$J_{12}\Phi^a = \frac{1}{2}\chi^a, \quad J_{22}\Phi^a = \omega^{ab}\chi_b, \quad J_{11}\Phi^a = 0. \quad (9.28b)$$

¹⁸It should be remarked that eq. (6.52) defines a trivial symmetry transformation involving the operator Δ .

The three generators J_{ij} may be naturally associated with the three Killing vectors $X_A = (J_A)^\mu{}_\nu \chi^\nu$, where $\chi^\mu = (g^{ab}\mathcal{K}_{\bar{b}}, g^{\bar{a}b}\mathcal{K}_b)$. We necessarily find that $\mathbf{S}^{ij} J_{ij} \Phi^a = -\mu \omega^{ab} \chi_b$ in accordance with (9.24). Note that for this choice of J_{ij} action, Φ^a must be on-shell since the operator Δ must be chosen to vanish.

Note the superconformal case is special since we have a triplet of Killing vectors to match the triplet of $SU(2)_R$ transformations; in the non-superconformal case we have only a single Killing vector, corresponding to the single $SO(2)_R$ transformation available.

9.3 An $SU(2)_R$ covariant geometric reformulation

Before moving on, we would like to discuss the reformulation of the $\mathcal{N} = 2$ superfield and constraints we have imposed in a way which is $SU(2)_R$ covariant à la Sierra and Townsend [55] (see also [47]).

We begin by taking Φ^μ to be a coordinate of a $4n$ -dimensional hyperkähler manifold with structure group $Sp(1) \times Sp(n)$. We use the index $i = \underline{1}, \underline{2}$ as the $Sp(1) \cong SU(2)$ index and $a = 1, \dots, 2n$ as the $Sp(n)$ index. Following [55] we introduce a vielbein f_μ^{ai} and its inverse f_{ai}^μ to convert between world-index vectors and tangent-space vectors. They obey the usual conditions

$$f_\mu^{ai} f_{ai}^\nu = \delta_\mu^\nu, \quad f_{ai}^\mu f_\mu^{bj} = \delta_a^b \delta_i^j. \quad (9.29)$$

In terms of these one can construct the metric $g_{\mu\nu}$ via

$$g_{\mu\nu} := f_\mu^{ai} f_\nu^{bj} \epsilon_{ij} \omega_{ab} \quad (9.30)$$

where ϵ_{ij} and ω_{ab} are the tangent space $Sp(1)$ and $Sp(n)$ metrics, respectively. In addition one can construct the covariantly constant complex structures

$$(J_A)^\mu{}_\nu := -i f_{ai}^\mu (\sigma_A)^i{}_j f_\nu^{aj} \quad (9.31)$$

which obey the quaternionic algebra

$$(J_A)^\mu{}_\nu (J_B)^\nu{}_\rho = -\delta_{AB} \delta^\mu{}_\rho + \epsilon_{ABC} (J_C)^\mu{}_\rho. \quad (9.32)$$

Finally, the $\mathcal{N} = 2$ superfield Φ^μ is assumed to obey the constraint

$$f_\mu^{a(i} \mathcal{D}_{\beta}^{j)} \Phi^\mu = f_\mu^{a(i} \bar{\mathcal{D}}_{\dot{\beta}}^{j)} \Phi^\mu = 0. \quad (9.33)$$

This implies the relation

$$\frac{1}{\sqrt{2}}\mathcal{D}_{\beta j}\Phi^\mu = f_{aj}{}^\mu\chi_\beta^a, \quad \frac{1}{\sqrt{2}}\bar{\mathcal{D}}_{\dot{\beta} j}\Phi^\mu = f_{aj}{}^\mu\bar{\chi}_{\dot{\beta}}^a \quad (9.34)$$

for Weyl fermion superfields χ_β^a and $\bar{\chi}_{\dot{\beta}}^a$, both carrying $\mathrm{Sp}(n)$ indices.

For the situation we considered in the previous subsection, there is a natural identification between the $\mathrm{Sp}(n)$ index a and half of the world indices $\mu = (a, \bar{a})$. The coordinate is $\Phi^\mu = (\Phi^a, \bar{\Phi}^{\bar{a}})$ and the corresponding vielbein $f_\mu{}^{bj}$ is given by

$$f_a{}^{b1} = 0, \quad f_a{}^{b2} = \delta_a{}^b, \quad f_{\bar{a}}{}^{b1} = i\omega_{\bar{a}}{}^b, \quad f_{\bar{a}}{}^{b2} = 0. \quad (9.35)$$

The metric is easily calculable and has the usual form. Similarly, the complex structures J_1 , J_2 , and J_3 are given by eqs. (6.48) and (6.49).

10 Supercurrents of the $\mathcal{N} = 2$ supersymmetric σ -model in AdS

We turn next to a brief discussion of the supercurrent of this $\mathcal{N} = 2$ nonlinear σ -model. In order to have a self-contained presentation, we discuss in the first subsection the purely $\mathcal{N} = 1$ supercurrent of the nonlinear σ -model. Then drawing upon our previous work on $\mathcal{N} = 2$ supercurrents, we construct in the second subsection the $\mathcal{N} = 2$ supercurrent associated with the model.

10.1 $\mathcal{N} = 1$ supercurrent in AdS

Recall that the most general nonlinear σ -model action involving only chiral superfields in AdS can be written

$$\begin{aligned} S &= \int d^4x d^4\theta E K + \int d^4x d^2\theta \mathcal{E} W + \int d^4x d^2\bar{\theta} \bar{\mathcal{E}} \bar{W} \\ &= \int d^4x d^4\theta E \mathcal{K}, \quad \mathcal{K} = K + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}}. \end{aligned} \quad (10.1)$$

We would like to discuss the supercurrent associated with this model.

As we showed in [56], the most general $\mathcal{N} = 1$ supercurrent multiplet in AdS is characterized by the conservation equation

$$\bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = \mathcal{D}_\alpha X - \frac{1}{4}\bar{\mathcal{D}}^2 \zeta_\alpha, \quad (10.2)$$

where $J_{\alpha\dot{\alpha}}$ is the supercurrent, and X and ζ_α the trace multiplets constrained by

$$\bar{\mathcal{D}}_{\dot{\alpha}}X = 0 , \quad \mathcal{D}_{(\alpha}\zeta_{\beta)} = 0 . \quad (10.3)$$

The case $\zeta_\alpha = 0$ corresponds to the Ferrara-Zumino multiplet which is associated with the old minimal formulation of AdS supergravity. On the other hand, the supercurrent with $X = 0$ corresponds to the non-minimal formulation of AdS supergravity [56].¹⁹ The specific feature of the AdS supersymmetry is that the trace multiplets are defined modulo a gauge transformation of the form

$$X \rightarrow X + \mu\Lambda , \quad \zeta_\alpha \rightarrow \zeta_\alpha + \mathcal{D}_\alpha\Lambda \quad (10.4)$$

for chiral Λ , $\bar{\mathcal{D}}_{\dot{\alpha}}\Lambda = 0$. This gauge symmetry allows one to set $X = 0$ and so the supercurrent (10.2) is completely *equivalent* to the non-minimal AdS supercurrent.

The general supercurrent (10.2) can be modified by an improvement transformation²⁰

$$J_{\alpha\dot{\alpha}} \rightarrow J_{\alpha\dot{\alpha}} + \mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot{\alpha}}\Upsilon - \bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha\Upsilon , \quad (10.5a)$$

$$X \rightarrow X + \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\Upsilon , \quad (10.5b)$$

$$\zeta_\alpha \rightarrow \zeta_\alpha - 2\mathcal{D}_\alpha(\bar{\Upsilon} + 2\Upsilon) , \quad (10.5c)$$

with Υ a well-defined complex scalar operator. The important feature of AdS superspace is that ζ_α can always be represented as a gradient,

$$\zeta_\alpha = \mathcal{D}_\alpha\zeta , \quad (10.6)$$

for some globally defined scalar operator ζ . As a result, the above improvement transformation allows us to choose

$$\zeta_\alpha = 0 . \quad (10.7)$$

In other words, the AdS supercurrent can be improved to a Ferrara-Zumino one. If the condition (10.7) holds, then the above improvement transformation reduces to

$$J_{\alpha\dot{\alpha}} \rightarrow J_{\alpha\dot{\alpha}} + 2i\mathcal{D}_{\alpha\dot{\alpha}}(\Psi - \bar{\Psi}) , \quad (10.8a)$$

$$X \rightarrow X + 4\mu\Psi + \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\bar{\Psi} , \quad (10.8b)$$

¹⁹As mentioned in [56], there is a certain freedom in choosing how to define the non-minimal supercurrent, leading to a one-parameter family of supercurrents. We show in Appendix D that these alternative supercurrents can easily be represented in the form (10.2).

²⁰If the gauge $X = 0$ is chosen, it is straightforward to modify the below improvement transformation to maintain this gauge.

for chiral Ψ , $\bar{\mathcal{D}}_{\dot{\alpha}}\Psi = 0$.

For the model (10.1) under consideration, the general AdS supercurrent (10.2) is

$$J_{\alpha\dot{\alpha}} = -\frac{1}{2}K_{a\bar{b}}\mathcal{D}_{\alpha}\phi^a\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\phi}^{\bar{b}} , \quad \zeta_{\alpha} = \mathcal{D}_{\alpha}K , \quad X = -W . \quad (10.9)$$

The gauge invariance (10.4) coincides with the Kähler transformation in AdS, which allows the supercurrent to be recast purely in the non-minimal form and in terms of the function \mathcal{K} alone:

$$J_{\alpha\dot{\alpha}} = -\frac{1}{2}\mathcal{K}_{a\bar{b}}\mathcal{D}_{\alpha}\phi^a\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\phi}^{\bar{b}} , \quad \zeta_{\alpha} = \mathcal{D}_{\alpha}\mathcal{K} . \quad (10.10)$$

For the same model, the Ferrara-Zumino supercurrent is given by

$$J_{\alpha\dot{\alpha}} = -\frac{1}{6}K_{a\bar{b}}\mathcal{D}_{\alpha}\phi^a\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\phi}^{\bar{b}} + \frac{i}{3}(K_a\mathcal{D}_{\alpha\dot{\alpha}}\phi^a - K_{\bar{a}}\mathcal{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\bar{a}}) \quad (10.11a)$$

$$X = \frac{1}{12}(\bar{\mathcal{D}}^2 - 4\mu)K - W . \quad (10.11b)$$

The Kähler transformation corresponds to a Ferrara-Zumino improvement transformation (10.8) for the choice $\Psi = F/6$, with F a holomorphic function of the target space coordinates $F = F(\phi)$. If we choose $F = W/\mu$, we find exactly

$$J'_{\alpha\dot{\alpha}} = -\frac{1}{6}\mathcal{K}_{a\bar{b}}\mathcal{D}_{\alpha}\phi^a\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\phi}^{\bar{b}} + \frac{i}{3}(\mathcal{K}_a\mathcal{D}_{\alpha\dot{\alpha}}\phi^a - \mathcal{K}_{\bar{a}}\mathcal{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\bar{a}}) \quad (10.12a)$$

$$X' = \frac{1}{12}(\bar{\mathcal{D}}^2 - 4\mu)\mathcal{K} \quad (10.12b)$$

with the supercurrent determined entirely by the function \mathcal{K} alone.

10.2 $\mathcal{N} = 2$ supercurrent in AdS

Let us now specialize to an $\mathcal{N} = 1$ Lagrangian \mathcal{K} which possesses $\mathcal{N} = 2$ supersymmetry in AdS. We would like to construct its supercurrent. Recall in [19] we showed that the natural supercurrent arising in $\mathcal{N} = 2$ supersymmetric theories AdS takes the form²¹

$$\frac{1}{4}(\bar{\mathcal{D}}_{ij} + 4\mathcal{S}_{ij})\mathcal{J} = w\mathbf{T}_{ij} - g_{ij}\mathcal{Y} \quad (10.13)$$

where \mathcal{J} is a real superfield corresponding to the $\mathcal{N} = 2$ supercurrent while \mathbf{T}_{ij} and \mathcal{Y} correspond to contributions to the $\mathcal{N} = 2$ trace multiplet. They obey

$$\mathcal{D}_{\alpha}^{(k}\mathbf{T}^{ij)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(k}\mathbf{T}^{ij)} = 0 , \quad (\mathbf{T}_{ij})^* = \mathbf{T}^{ij} \quad (10.14)$$

$$\mathcal{D}_i^{\dot{\alpha}}\mathcal{Y} = 0 , \quad \frac{1}{4}(\mathcal{D}^{ij} + 4\mathcal{S}^{ij})\mathcal{Y} = \frac{1}{4}(\bar{\mathcal{D}}^{ij} + 4\bar{\mathcal{S}}^{ij})\bar{\mathcal{Y}} . \quad (10.15)$$

²¹In the rigid supersymmetric limit, the supercurrent (10.13) reduces to that constructed in [57].

The first condition says that \mathbf{T}_{ij} is an $\mathcal{N} = 2$ linear multiplet; the second condition says that \mathbf{Y} is an $\mathcal{N} = 2$ reduced chiral multiplet. The constant parameters w and $g_{ij} = \overline{g^{i\bar{j}}}$ in (10.13) obey

$$w\bar{w} = \sqrt{\frac{1}{2}g^{ij}g_{ij}} = 1, \quad g_{ij} \propto \mathbf{S}_{ij}.$$

These parameters can always be chosen so that the supercurrent equation takes the simpler form

$$\frac{1}{4}(\bar{\mathcal{D}}_{ij} + 4\mathbf{S}_{ij})\mathcal{J} = \mathbf{T}_{ij} - \frac{\mathbf{S}_{ij}}{\mathbf{S}}\mathbf{Y}. \quad (10.16)$$

One can derive the $\mathcal{N} = 1$ supercurrent from the $\mathcal{N} = 2$ supercurrent. Within AdS, this requires the choice $\mathbf{S}_{12} = 0$. Then the $\mathcal{N} = 1$ supercurrent takes the Ferrara-Zumino form with

$$J_{\alpha\dot{\alpha}} = \frac{1}{4}[\mathcal{D}_{\alpha}^2, \bar{\mathcal{D}}_{\dot{\alpha}2}]\mathcal{J}| - \frac{1}{12}[\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}}]\mathcal{J}| \quad (10.17)$$

where $|$ denotes projection to $\mathcal{N} = 1$. The corresponding trace multiplet turns out to be

$$X = \frac{1}{3}\mathbf{T}_{11}| - \frac{2}{3}\frac{\mu}{|\mu|}\mathbf{Y}|. \quad (10.18)$$

The $\mathcal{N} = 2$ analogue of the improvement transformation (10.8) is

$$\mathcal{J} \rightarrow \mathcal{J} + \mathcal{R} + \bar{\mathcal{R}} \quad (10.19a)$$

$$\mathbf{T}_{ij} \rightarrow \mathbf{T}_{ij} + \frac{1}{4}(\bar{\mathcal{D}}_{ij} + 4\mathbf{S}_{ij})\bar{\mathcal{R}} \quad (10.19b)$$

$$\mathbf{Y} \rightarrow \mathbf{Y} - \mathbf{S}\mathcal{R} \quad (10.19c)$$

where \mathcal{R} is a reduced chiral superfield. This transformation allows us to eliminate \mathbf{Y} . It is an easy exercise to check that this leads to the $\mathcal{N} = 1$ improvement transformation (10.8) with $F = \mathcal{R}|$.

Now we would like to postulate the form of the $\mathcal{N} = 2$ supercurrent for the nonlinear σ -model in AdS. Because there is a single function \mathcal{K} that parametrizes the $\mathcal{N} = 1$ action, we will make a guess that

$$\mathcal{J} = -\frac{1}{2}\mathcal{K} \quad (10.20)$$

and examine whether this is reasonable. First by explicit calculation, one can check that on-shell

$$\frac{1}{4}(\bar{\mathcal{D}}_{ij} + 4\mathbf{S}_{ij})\mathcal{J} = -\frac{\mathbf{S}_{ij}}{\mathbf{S}}\mathbf{Y} \quad (10.21)$$

where

$$\mathcal{Y} = -\frac{|\mu|}{2}\mathcal{K}_a g^{a\bar{b}}\mathcal{K}_{\bar{b}} + \frac{|\mu|}{2}\mathcal{K} - \frac{\bar{\mu}}{8|\mu|}\nabla_{\bar{a}}\mathcal{K}_{\bar{b}}\bar{\mathcal{D}}_{\dot{\alpha}\perp}\bar{\Phi}^{\dot{\alpha}}\bar{\mathcal{D}}_{\perp}^{\dot{\alpha}}\bar{\Phi}^{\bar{b}} \quad (10.22)$$

is a reduced chiral superfield on-shell. We may provide further justification of this $\mathcal{N} = 2$ supercurrent by considering its $\mathcal{N} = 1$ reduction. The result is exactly (10.12).

11 Concluding remarks

We have covered a great deal of ground, so let us briefly recap the main results of this work. Our focus in sections 2 through 4 was $\mathcal{N} = 2$ tensor multiplet models. The key result there was that within an AdS background the $\mathcal{N} = 1$ Lagrangian $L(\varphi^I, \bar{\varphi}^I, G^I)$ must obey not only the usual Laplace equation

$$\frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} + \frac{\partial^2 L}{\partial G^I \partial G^J} = 0 \quad (11.1)$$

but also an additional constraint

$$\text{Re} \left(\mu \frac{\partial L}{\partial \varphi^I} - \mu \frac{\partial^2 L}{\partial \varphi^I \partial G^J} G^J - 2\mu \frac{\partial^2 L}{\partial \varphi^I \partial \bar{\varphi}^J} \bar{\varphi}^J \right) = 0 \quad (11.2)$$

arising from the requirement that the models respect the $\text{SO}(2)$ invariance of AdS. It was shown long ago [13] how the first condition finds its solution most naturally expressed in the language of $\mathcal{N} = 2$ projective superspace. We have briefly discussed in section 5 how the second constraint also emerges naturally in the same setting and moreover may easily be understood by requiring that the AdS Lagrangian arise from a superconformal tensor model where one of the tensor multiplets has been “frozen.” Such an analysis was anticipated in [26].

These tensor multiplet models are dual to a subclass of nonlinear σ -models – those with n Abelian isometries. In sections 6 and 7 we discussed the properties of the most general nonlinear σ -models and uncovered a number of fascinating features, which were previously reported in [44].

First, the supersymmetry algebra closes off-shell. This is a new type of structure that has no analogue in Minkowski space. Indeed, in order to have off-shell supersymmetry for general $\mathcal{N} = 2$ nonlinear σ -models in Minkowski space, one has to use the harmonic [11, 12] or the projective [13, 14] superspace approaches in which the off-shell hypermultiplet realizations involve an infinite number of auxiliary fields. In our construction, the hypermultiplet is described using a minimal realization of two

ordinary $\mathcal{N} = 1$ chiral superfields with $8 + 8$ degrees of freedom. Off-shell supersymmetry is also characteristic of the gauge models for massless higher spin $\mathcal{N} = 2$ supermultiplets in AdS constructed in [41] using $\mathcal{N} = 1$ superfields. Since those theories are linearized, one may argue that their off-shell supersymmetry is not really impressive. However, now we have demonstrated that the formulation of the most general nonlinear $\mathcal{N} = 2$ supersymmetric σ -models in terms of $\mathcal{N} = 1$ chiral superfields is also off-shell. This gives us some evidence to believe that, say, general $\mathcal{N} = 2$ super Yang-Mills theories in AdS possess an off-shell formulation in which the hypermultiplet is realized in terms of two chiral superfields. If this conjecture is correct, there may be nontrivial implications for quantum effective actions.

The second intriguing feature is that the target space geometry must be hyperkähler with a special Killing vector V^μ which rotates the complex structures

$$\mathcal{L}_V J_1 = J_3 \sin \theta, \quad \mathcal{L}_V J_2 = -J_3 \cos \theta, \quad \mathcal{L}_V J_3 = J_2 \cos \theta - J_1 \sin \theta \quad (11.3)$$

where $\theta := \arg \mu$. It necessarily leaves invariant one linear combination of complex structures, which we denote J_{AdS} . There is an underlying physical reason for this: $\mathcal{N} = 2$ AdS supersymmetry requires an $\text{SO}(2)$ isometry on the target space as well as on the covariant derivatives. Most importantly, in the coordinates where J_{AdS} is diagonal, V^μ is holomorphic and is associated with a Killing potential \mathcal{K} via

$$V^\mu = \frac{1}{2} J_{\text{AdS}}{}^\mu{}_\nu \nabla^\nu \mathcal{K}. \quad (11.4)$$

In the usual coordinates where J_3 is diagonal, \mathcal{K} is the Kähler potential and the AdS Lagrangian for the nonlinear σ -model. This discussion shows which hyperkähler manifolds can be used as target spaces of $\mathcal{N} = 2$ nonlinear σ -models in AdS and gives the procedure for constructing the σ -model action.

In section 8, we discussed $\mathcal{N} = 2$ superconformal σ -models for which the target spaces are hyperkähler cones. In this class, the additional $\text{SO}(2)$ isometry required by AdS is naturally realized within a larger $\text{SU}(2)$ isometry group required by the $\mathcal{N} = 2$ superconformal algebra. Unlike the situation in AdS, however, the full $\mathcal{N} = 2$ superconformal algebra closes only *on-shell*. This demonstrates the particular importance of AdS for off-shell closure.

In section 9, we described a simple $\mathcal{N} = 2$ superfield formulation within AdS which reproduces all of the features of the $\mathcal{N} = 1$ model and explains their off-shell closure. By allowing the behavior of the $\text{SO}(2)$ generator to be dictated by algebraic consistency rather than target space geometry, it can be made to perform the same function in AdS that a central charge does in Minkowski. This formulation also makes

clear why the $\mathcal{N} = 2$ superconformal algebra does not close. Finally, in section 10, we discussed briefly the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supercurrents of nonlinear σ -models in AdS. Such supercurrents have been of interest recently [58, 59, 60, 46, 61, 62, 63, 64, 65, 66, 44].

There remain a number of open questions. The most apparent lies in the contrast between our understanding of tensor models and that of general σ -models. For the former, the projective superspace approach elegantly solves the problem with a minimum of technical difficulty. In particular, it clearly explains how the geometry of tensor models in AdS can be understood as a superconformal tensor model with an additional “frozen” tensor multiplet. For general σ -models, we are led to ask two corresponding questions. First, can we think of general σ -models in AdS as arising from a frozen superconformal model? Second, can we use the projective superspace formulation of [26] involving polar multiplets to gain insight about the geometric features of general σ -models? In practice, this is a daunting task requiring the elimination of an infinite number of auxiliary fields. In Minkowski space, this problem was solved for a large class of $\mathcal{N} = 2$ supersymmetric σ -models on cotangent bundles of Hermitian symmetric spaces [67, 68, 69, 70, 71, 72]. We expect these results can be generalized to the AdS case.

Another open question is how the selection of the $\text{SO}(2)$ isometry reflects the choice of which supersymmetry is made manifest. Recently, work on nonlinear σ -models in AdS_5 with eight supercharges [50, 51] has uncovered a similar $\text{SO}(2)$ isometry in a different context. In these works, which are inspired by brane world scenarios, the AdS_5 geometry is foliated with flat four dimensional hypersurfaces perpendicular to the fifth dimension with metric

$$ds^2 = e^{-2\lambda z} \eta_{mn} dx^m dx^n + dz^2 . \quad (11.5)$$

Nonlinear σ -models are then constructed from *flat* 4D $\mathcal{N} = 1$ chiral superfields parametrically dependent on z . The authors found that the target space manifold must not only be hyperkähler but also be equipped with a certain Killing vector which acts as an $\text{SO}(2)$ rotation on the complex structures. The main difference from our result is that the Killing vector is *manifestly* holomorphic – that is, holomorphic with respect to J_3 .

The reason for this difference can be explained as follows. As shown in [73], the 5D $\mathcal{N} = 1$ AdS superspace, $\text{AdS}^{5|8}$, is formulated in terms of a constant isotriplet \mathbf{S}^{ij} , which can be interpreted as the constant torsion tensor of $\text{AdS}^{5|8}$; the same tensor \mathbf{S}^{ij} occurs in the case of $\text{AdS}^{4|8}$. The foliation (11.5), which was recently employed in

[50, 51], was used (in slightly different form) several years ago in [73] to realize $\text{AdS}^{5|8}$ as a conformally flat 5D superspace with flat 4D $\mathcal{N} = 1$ subspaces. In [73] the above foliation arose naturally upon choosing an $\text{SU}(2)$ gauge where $\mathbf{S}^{11} = \mathbf{S}^{22} = 0$.

What about $\text{AdS}^{4|8}$? Again one has a constant isotriplet \mathbf{S}^{ij} . The choice $\mathbf{S}^{12} = 0$ allows one to naturally select an $\text{AdS}^{4|4}$ subspace of $\text{AdS}^{4|8}$ simply by turning off the second set of Grassmann coordinates [73].²² This is the choice we have made in this work, and it should come as no surprise that the allowed target space geometry differs from that of [50, 51] simply by a different choice of which complex structure is manifest. Instead of this choice, we could take $\mathbf{S}^{11} = \mathbf{S}^{22} = 0$ in $\text{AdS}^{4|8}$ by applying a rigid $\text{SU}(2)$ transformation. (The same $\text{SU}(2)$ which acts on \mathbf{S}^{ij} should act on the two-sphere of complex structures.) In analogy to five dimensions, we expect that this choice be naturally respected by a foliation involving 3D $\mathcal{N} = 2$ Minkowski superspace. The allowed target space geometries should again be hyperkähler manifolds with a special $\text{SO}(2)$ Killing vector but with the preferred complex structure matching the manifest one as in [50, 51]. It would be interesting to see this borne out by an explicit construction.

Acknowledgements:

This work is supported in part by the Australian Research Council and by a UWA Research Development Award. D.B. would like to thank Jon Bagger for correspondence.

A $\mathcal{N} = 1$ (superconformal) Killing vector fields

In this appendix we discuss superconformal and isometry transformations of the $\mathcal{N} = 1$ AdS superspace, $\text{AdS}^{4|4}$, which is a maximally symmetric solution of old minimal supergravity with a cosmological term (see [38] for more details regarding the $\mathcal{N} = 1$ AdS supergeometry). The corresponding covariant derivatives,²³

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A{}^M \partial_M + \frac{1}{2} \phi_A{}^{bc} M_{bc} \equiv E_A + \phi_A, \quad (\text{A.1})$$

²²See for example the discussion in Appendix B.2.

²³We follow the notation and conventions adopted in [38], with the only exception that we use lower case *Roman* letters for tangent-space vector indices.

obey the following (anti-)commutation relations:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{\mu}M_{\alpha\beta} , \quad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 4\mu\bar{M}_{\dot{\alpha}\dot{\beta}} , \quad (\text{A.2a})$$

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\beta}}\} = -2i(\sigma^c)_{\alpha\dot{\beta}}\mathcal{D}_c \equiv -2i\mathcal{D}_{\alpha\dot{\beta}} , \quad (\text{A.2b})$$

$$[\mathcal{D}_a, \mathcal{D}_\beta] = -\frac{i}{2}\bar{\mu}(\sigma_a)_{\beta\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\gamma}} , \quad [\mathcal{D}_a, \bar{\mathcal{D}}_{\dot{\beta}}] = \frac{i}{2}\mu(\sigma_a)_{\gamma\dot{\beta}}\mathcal{D}^\gamma , \quad (\text{A.2c})$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -|\mu|^2 M_{ab} , \quad (\text{A.2d})$$

with μ a complex non-vanishing parameter. One can think of μ as a square root of the curvature of the anti-de Sitter space. The Lorentz generators with vector indices ($M_{ab} = -M_{ba}$) and spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$) are related to each other by the rule:

$$M_{ab} = (\sigma_{ab})^{\alpha\beta}M_{\alpha\beta} - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}} , \quad M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab} , \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}M_{ab} .$$

The Lorentz generators act on the spinor covariant derivative by the rule:

$$[M_{\alpha\beta}, \mathcal{D}_\gamma] = \frac{1}{2}(\varepsilon_{\gamma\alpha}\mathcal{D}_\beta + \varepsilon_{\gamma\beta}\mathcal{D}_\alpha) , \quad [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{\mathcal{D}}_{\dot{\gamma}}] = \frac{1}{2}(\varepsilon_{\dot{\gamma}\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta}} + \varepsilon_{\dot{\gamma}\dot{\beta}}\bar{\mathcal{D}}_{\dot{\alpha}}) , \quad (\text{A.3})$$

while $[M_{\alpha\beta}, \bar{\mathcal{D}}_{\dot{\gamma}}] = [\bar{M}_{\dot{\alpha}\dot{\beta}}, \mathcal{D}_\gamma] = 0$.

In accordance with [38], a real vector field, $\xi^A = (\xi^a, \xi^\alpha, \bar{\xi}_{\dot{\alpha}})$, on $\text{AdS}^{4|4}$ is said to be superconformal Killing if the corresponding infinitesimal coordinate transformation can be accompanied by special Lorentz and super-Weyl transformations such that the covariant derivatives remain unchanged. In terms of the first-order differential operator

$$\xi := \xi^a\mathcal{D}_a + \xi^\alpha\mathcal{D}_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} , \quad (\text{A.4})$$

the above definition is equivalent to the condition

$$[\xi + \frac{1}{2}\lambda^{cd}M_{cd}, \mathcal{D}_\alpha] + (\bar{\sigma} - \frac{1}{2}\sigma)\mathcal{D}_\alpha + (\mathcal{D}^\beta\sigma)M_{\alpha\beta} = 0 . \quad (\text{A.5})$$

This same transformation acts on any *primary* tensor superfield U as

$$\delta U = -\xi U - \frac{1}{2}\lambda^{cd}M_{cd}U - \frac{\Delta}{2}(\sigma + \bar{\sigma})U - \frac{3w}{4}(\sigma - \bar{\sigma})U \quad (\text{A.6})$$

where Δ is the conformal dimension of U and w is its chiral weight. If $U = \Phi$ is chiral, then we must have $2\Delta = 3w$ and the above expression can be rewritten

$$\delta\Phi = -\xi\Phi - \frac{1}{2}\lambda^{cd}M_{cd}\Phi - \sigma\Delta\Phi \quad (\text{A.7})$$

Let us now solve for the relations that the $\mathcal{N} = 1$ AdS parameter must obey. Making use of the (anti-)commutation relations (A.2) gives

$$\mathcal{D}_\alpha \xi^b - 2i(\sigma^b)_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} = 0 , \quad (\text{A.8a})$$

$$-\frac{i}{2}\bar{\mu}\xi^b(\sigma_b)_{\alpha\dot{\beta}} - \mathcal{D}_\alpha \bar{\xi}_{\dot{\beta}} = 0 , \quad (\text{A.8b})$$

$$\mathcal{D}_\alpha \xi^\beta - \lambda_\alpha{}^\beta - \delta_\alpha{}^\beta (\bar{\sigma} - \frac{1}{2}\sigma) = 0 , \quad (\text{A.8c})$$

$$\mathcal{D}_\alpha \bar{\lambda}^{\dot{\beta}\gamma} = 0 , \quad (\text{A.8d})$$

$$2\bar{\mu}(\delta_\alpha{}^\beta \xi^\gamma + \delta_\alpha{}^\gamma \xi^\beta) + \mathcal{D}_\alpha \lambda^{\beta\gamma} - \frac{1}{2}(\delta_\alpha{}^\beta \mathcal{D}^\gamma \sigma + \delta_\alpha{}^\gamma \mathcal{D}^\beta \sigma) = 0 . \quad (\text{A.8e})$$

Just as in a Minkowski background [38], this set of equations can be solved in terms of the single parameter $\xi_{\alpha\dot{\alpha}}$ obeying the so-called “master equation”

$$\mathcal{D}_{(\beta} \xi_{\alpha)\dot{\alpha}} = \bar{\mathcal{D}}^{(\dot{\beta}} \xi^{\dot{\alpha})\alpha} = 0 . \quad (\text{A.9})$$

Note that this equation implies

$$\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = \frac{1}{2} \eta_{ab} \mathcal{D}_c \xi^c \quad (\text{A.10})$$

which guarantees that the lowest component of ξ^a is a conformal Killing vector.

The other parameters are given by

$$\xi_\alpha = \frac{i}{8} \bar{\mathcal{D}}^{\dot{\beta}} \xi_{\alpha\dot{\beta}} , \quad \lambda_{ab} = \mathcal{D}_{[a} \xi_{b]} , \quad (\text{A.11})$$

$$\sigma = -\frac{i}{24} \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \xi^{\dot{\alpha}\alpha} - \frac{i}{12} \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_\alpha \xi^{\dot{\alpha}\alpha} . \quad (\text{A.12})$$

By construction, the super-Weyl parameter σ must be chiral,

$$\bar{\mathcal{D}}_{\dot{\alpha}} \sigma = 0 . \quad (\text{A.13})$$

This property follows from the relation (A.12) and the master equation (A.9). We note that the dilatation and chiral rotation parameters are given respectively by

$$\text{Re } \sigma = -\frac{1}{8} \mathcal{D}_{\alpha\dot{\alpha}} \xi^{\dot{\alpha}\alpha} = \frac{1}{4} \mathcal{D}_a \xi^a , \quad (\text{A.14})$$

$$\frac{3}{2} \text{Im } \sigma = \frac{1}{32} [\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] \xi^{\dot{\alpha}\alpha} . \quad (\text{A.15})$$

It can be shown that the superconformal Killing vector fields generate the $\mathcal{N} = 1$ superconformal group $\text{SU}(2, 2|1)$.

Any superconformal Killing vector field ξ with the additional property

$$\sigma = 0 \implies \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \xi^{\dot{\alpha}\alpha} = 0 \quad (\text{A.16})$$

is called a Killing vector field of the $\mathcal{N} = 1$ AdS superspace. All Killing vector fields are characterized by the property

$$[\xi + \frac{1}{2}\lambda^{\text{cd}}M_{\text{cd}}, \mathcal{D}_A] = 0 . \quad (\text{A.17})$$

This master equation implies the relations

$$\mathcal{D}_{(\alpha}\xi_{\beta)\dot{\beta}} = 0 , \quad \bar{\mathcal{D}}^{\dot{\beta}}\xi_{\alpha\dot{\beta}} + 8i\xi_{\alpha} = 0 , \quad (\text{A.18a})$$

$$\mathcal{D}_{\alpha}\xi^{\alpha} = 0 , \quad \bar{\mathcal{D}}_{\dot{\alpha}}\xi_{\alpha} + \frac{i}{2}\mu\xi_{\alpha\dot{\alpha}} = 0 , \quad (\text{A.18b})$$

$$\lambda_{\alpha\beta} = \mathcal{D}_{\alpha}\xi_{\beta} , \quad (\text{A.18c})$$

and these equations follow from (A.11) and (A.9) using the additional constraint (A.16). Since (A.16) implies $\mathcal{D}_a\xi^a = 0$, it follows that the conformal Killing equation (A.10) reduces to

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = 0 . \quad (\text{A.19})$$

The AdS Killing vector fields generate the isometry group of the $\mathcal{N} = 1$ AdS superspace, $\text{OSp}(1|4)$.

B $\mathcal{N} = 2$ (superconformal) Killing vector fields

The four-dimensional $\mathcal{N} = 2$ AdS superspace

$$\text{AdS}^{4|8} := \frac{\text{OSp}(2|4)}{\text{SO}(3,1) \times \text{SO}(2)}$$

can be realized as a maximally symmetric geometry that originates within the superspace formulation of $\mathcal{N} = 2$ conformal supergravity developed in [27]. Assuming the superspace is parametrized by local bosonic (x) and fermionic ($\theta, \bar{\theta}$) coordinates $\mathbf{z}^{\mathcal{M}} = (x^m, \theta_i^{\mu}, \bar{\theta}_{\dot{\mu}}^i)$ (where $m = 0, 1, \dots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $i = \underline{1}, \underline{2}$), the corresponding covariant derivatives

$$\mathcal{D}_{\mathcal{A}} = (\mathcal{D}_a, \mathcal{D}_{\alpha}^i, \bar{\mathcal{D}}_{\dot{\alpha}}^i) = E_{\mathcal{A}}^{\mathcal{M}}\partial_{\mathcal{M}} + \frac{1}{2}\phi_{\mathcal{A}}^{\text{bc}}M_{\text{bc}} + \phi_{\mathcal{A}}\mathbf{S}^{ij}J_{ij} , \quad i, j = \underline{1}, \underline{2} \quad (\text{B.1})$$

obey the algebra [27, 26]:

$$\{\mathcal{D}_{\alpha}^i, \mathcal{D}_{\beta}^j\} = 4\mathbf{S}^{ij}M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathbf{S}^{kl}J_{kl} , \quad \{\mathcal{D}_{\alpha}^i, \bar{\mathcal{D}}_{\dot{\alpha}j}\} = -2i\delta_j^i\mathcal{D}_{\alpha\dot{\alpha}} , \quad (\text{B.2a})$$

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta}^i] = -i\varepsilon_{\alpha\beta}\mathbf{S}^{ij}\bar{\mathcal{D}}_{\dot{\alpha}j} , \quad [\mathcal{D}_a, \mathcal{D}_b] = -\mathbf{S}^2M_{ab} , \quad (\text{B.2b})$$

where \mathbf{S}^{ij} is a *covariantly constant* and *constant* real iso-triplet, $\mathbf{S}^{ji} = \mathbf{S}^{ij}$, $\overline{\mathbf{S}^{ij}} = \mathbf{S}_{ij} = \varepsilon_{ik}\varepsilon_{jl}\mathbf{S}^{kl}$, and $\mathbf{S}^2 := \frac{1}{2}\mathbf{S}^{ij}\mathbf{S}_{ij}$. The SU(2) generators, J_{kl} , act on the spinor covariant derivatives by the rule:

$$[J_{kl}, \mathcal{D}_\alpha^i] = -\frac{1}{2}(\delta_k^i \mathcal{D}_{\alpha l} + \delta_l^i \mathcal{D}_{\alpha k}) . \quad (\text{B.3})$$

This superspace proves to be a conformally flat solution to the equations of motion for $\mathcal{N} = 2$ supergravity with a cosmological term [19].

B.1 $\mathcal{N} = 2$ superconformal Killing vector fields

We now turn to studying the algebra of superconformal Killing vector fields of $\text{AdS}^{4|8}$. By definition, a real vector field, $\xi^A = (\xi^a, \xi_i^\alpha, \bar{\xi}_{\dot{\alpha}}^i)$, on $\text{AdS}^{4|8}$ is said to be superconformal Killing if the corresponding infinitesimal coordinate transformation can be accompanied by special local Lorentz, local SU(2) and super-Weyl transformations such that the covariant derivatives remain unchanged,

$$[\xi + \frac{1}{2}\lambda^{\text{cd}}M_{\text{cd}} + \lambda^{kl}J_{kl}, \mathcal{D}_\alpha^i] + \frac{1}{2}\bar{\sigma}\mathcal{D}_\alpha^i + (\mathcal{D}^{\beta i}\sigma)M_{\alpha\beta} - (\mathcal{D}_{\alpha j}\sigma)J^{ij} = 0 , \quad (\text{B.4})$$

where

$$\xi := \xi^a \mathcal{D}_a + \xi_i^\alpha \mathcal{D}_\alpha^i + \bar{\xi}_{\dot{\alpha}}^i \bar{\mathcal{D}}_{\dot{\alpha}}^i . \quad (\text{B.5})$$

This same transformation acts on any $\mathcal{N} = 2$ *primary* tensor superfield \mathbf{U} as

$$\delta \mathbf{U} = -\xi \mathbf{U} - \frac{1}{2}\lambda^{\text{cd}}M_{\text{cd}}\mathbf{U} - \lambda^{jk}J_{jk}\mathbf{U} - \frac{\Delta}{2}(\sigma + \bar{\sigma})\mathbf{U} - \frac{w}{4}(\sigma - \bar{\sigma})\mathbf{U} \quad (\text{B.6})$$

where Δ is the conformal dimension of \mathbf{U} and w is its chiral weight. If $\mathbf{U} = \Phi$ is chiral, then we must have $2\Delta = w$ and the above expression can be rewritten

$$\delta \Phi = -\xi \Phi - \frac{1}{2}\lambda^{\text{cd}}M_{\text{cd}}\Phi - \lambda^{jk}J_{jk}\Phi - \sigma \Delta \Phi \quad (\text{B.7})$$

It follows from (B.4) that

$$\mathcal{D}_\alpha^i \xi^b - 2i(\sigma^b)_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}i} = 0 , \quad (\text{B.8a})$$

$$\frac{i}{2}\mathbf{S}^{ij}\xi_{\alpha\dot{\beta}} - \mathcal{D}_\alpha^i \bar{\xi}_{\dot{\beta}}^j = 0 , \quad (\text{B.8b})$$

$$-\mathcal{D}_\alpha^i \xi_j^\beta + \delta_j^i \lambda_\alpha^\beta + \delta_\alpha^\beta \lambda_j^i + \frac{1}{2}\delta_\alpha^\beta \delta_j^i \bar{\sigma} = 0 , \quad (\text{B.8c})$$

$$\mathcal{D}_\alpha^i \bar{\lambda}^{\dot{\beta}\gamma} = 0 , \quad (\text{B.8d})$$

$$2\mathbf{S}^{ij}(\delta_\alpha^\beta \xi^\gamma + \delta_\alpha^\gamma \xi^\beta) - \mathcal{D}_\alpha^i \lambda^{\beta\gamma} + \frac{1}{2}(\delta_\alpha^\beta \mathcal{D}^{\gamma i} \sigma + \delta_\alpha^\gamma \mathcal{D}^{\beta i} \sigma) = 0 , \quad (\text{B.8e})$$

$$2\xi_\alpha^i S^{kl} - \mathcal{D}_\alpha^i \lambda^{kl} + \frac{1}{2}(\varepsilon^{ik} \mathcal{D}_\alpha^l \sigma + \varepsilon^{il} \mathcal{D}_\alpha^k \sigma) = 0 . \quad (\text{B.8f})$$

As in the $\mathcal{N} = 1$ case, all the parameters above can be expressed in terms of the single parameter $\xi_{\alpha\dot{\alpha}}$, which obeys the master equation

$$\mathcal{D}_{(\beta}^i \xi_{\alpha)\dot{\alpha}} = \bar{\mathcal{D}}_j^{(\dot{\beta}} \xi^{\dot{\alpha})\alpha} = 0 . \quad (\text{B.9})$$

The other parameters are given by

$$\xi_{\alpha i} = \frac{i}{8} \bar{\mathcal{D}}_i^{\dot{\beta}} \xi_{\alpha\dot{\beta}} , \quad \lambda_{ab} = \mathcal{D}_{[a} \xi_{b]} , \quad (\text{B.10})$$

$$\lambda^{ij} = -\frac{i}{16} \mathcal{D}_{\alpha}^{(i} \bar{\mathcal{D}}_{\dot{\alpha}}^{j)} \xi^{\dot{\alpha}\alpha} , \quad \sigma = -\frac{i}{16} \bar{\mathcal{D}}_{\dot{\alpha}j} \mathcal{D}_{\alpha}^j \xi^{\dot{\alpha}\alpha} \quad (\text{B.11})$$

In accordance with [27], the super-Weyl parameter σ must be chiral,

$$\bar{\mathcal{D}}_{\dot{\alpha}i} \sigma = 0 . \quad (\text{B.12})$$

This property follows from the master equation (B.9). Note that the dilatation and U(1) parameters are given respectively by

$$\text{Re } \sigma = -\frac{1}{8} \mathcal{D}_{\alpha\dot{\alpha}} \xi^{\dot{\alpha}\alpha} = \frac{1}{4} \mathcal{D}_a \xi^a , \quad (\text{B.13})$$

$$\frac{1}{2} \text{Im } \sigma = -\frac{1}{64} [\mathcal{D}_{\alpha}^j, \bar{\mathcal{D}}_{\dot{\alpha}j}] \xi^{\dot{\alpha}\alpha} . \quad (\text{B.14})$$

The superconformal equations (B.8) have a number of implications of which we now list only a few, the most important for our subsequent analysis. From eq. (B.8f) we deduce

$$2\xi_{\alpha}^{(i} S^{kl)} - \mathcal{D}_{\alpha}^{(i} \lambda^{kl)} = 0 . \quad (\text{B.15})$$

In conjunction with the first relation in (B.11), this leads to

$$\mathcal{D}^{\alpha(i} \mathcal{D}_{\alpha}^j \lambda^{kl)} + 4S^{(kl} \lambda^{ij)} = 0 . \quad (\text{B.16})$$

Again from (B.8f) we derive

$$4S^2 \xi_{\alpha i} - \varepsilon_{ij} \mathcal{D}_{\alpha}^j \lambda^{kl} S_{kl} + S_{ik} \mathcal{D}_{\alpha}^k \sigma = 0 . \quad (\text{B.17})$$

Each of these can be checked against the explicit solutions given above in terms of the parameter $\xi_{\alpha\dot{\alpha}}$ obeying the master equation (B.9).

The above analysis is a natural extension of that given in the $\mathcal{N} = 2$ super-Poincaré case in [74] (see also [75]).

The superconformal Killing vector fields of the $\mathcal{N} = 2$ AdS superspace, $\text{AdS}^{4|8}$, prove to generate the supergroup $\text{SU}(2, 2|2)$, which is also the superconformal group of the $\mathcal{N} = 2$ Minkowski superspace, $\mathbb{R}^{4|8}$. The superspaces $\text{AdS}^{4|8}$ and $\mathbb{R}^{4|8}$ have the same superconformal group, since they are conformally related or, equivalently, since $\text{AdS}^{4|8}$ is conformally flat (see, e.g., [26, 76]).

B.2 $\mathcal{N} = 1$ reduction

By applying a rigid $SU(2)$ rotation to the covariant derivatives, it is always possible to choose the iso-triplet \mathbf{S}^{ij} such that

$$\mathbf{S}^{12} = 0 . \quad (\text{B.18})$$

We denote the non-vanishing components of \mathbf{S}^{ij} as

$$\mathbf{S}^{11} = \mathbf{S}_{22} = -\bar{\mu} , \quad \mathbf{S}^{22} = \mathbf{S}_{11} = -\mu . \quad (\text{B.19})$$

With the choice (B.18) made, the above relations imply the following conditions:

$$\mathcal{D}_\alpha^1 \bar{\xi}_\beta^2 = 0 \implies \bar{\mathcal{D}}_{\dot{\alpha}1} \xi_2^\beta = 0 , \quad (\text{B.20a})$$

$$[(\mathcal{D}^1)^2 - 4\bar{\mu}] \lambda^{11} = 0 \implies [(\bar{\mathcal{D}}_1)^2 - 4\mu] \lambda^{22} = 0 , \quad (\text{B.20b})$$

$$\mathcal{D}_\alpha^1 (\lambda^{12} - \frac{1}{2} \sigma) = 0 \implies \bar{\mathcal{D}}_{\dot{\alpha}1} (\lambda^{12} + \frac{1}{2} \bar{\sigma}) = 0 . \quad (\text{B.20c})$$

From (B.15) we derive

$$\xi_{\alpha\bar{2}} = -\frac{\mu}{2|\mu|^2} \mathcal{D}_\alpha^1 \lambda^{11} \implies \bar{\xi}_{\dot{\alpha}}^2 = -\frac{\bar{\mu}}{2|\mu|^2} \bar{\mathcal{D}}_{\dot{\alpha}1} \lambda^{22} . \quad (\text{B.21})$$

Then eq. (B.20a) tells us that

$$\bar{\mathcal{D}}_{\dot{\alpha}1} \mathcal{D}_\alpha^1 \lambda^{11} = 0 \implies \mathcal{D}_\alpha^1 \bar{\mathcal{D}}_{\dot{\alpha}1} \lambda^{22} = 0 . \quad (\text{B.22})$$

Eq. (B.17) also implies

$$4|\mu|^2 \xi_{\alpha\bar{2}} - \mathcal{D}_\alpha^1 (\mu \lambda^{11} + \bar{\mu} \lambda^{22}) - \bar{\mu} \mathcal{D}_\alpha^2 \sigma = 0 . \quad (\text{B.23})$$

We are interested in an $\mathcal{N} = 1$ AdS reduction of the $\mathcal{N} = 2$ superconformal Killing vector fields. In other words, we would like to derive those transformations in $\mathcal{N} = 1$ AdS superspace which are generated by an arbitrary $\mathcal{N} = 2$ superconformal Killing vector field.²⁴ Given a tensor superfield $\mathbf{U}(x, \theta_i, \bar{\theta}^i)$ in $\mathcal{N} = 2$ AdS superspace, we introduce its $\mathcal{N} = 1$ projection

$$U = \mathbf{U}| := \mathbf{U}(x, \theta_i, \bar{\theta}^i)|_{\theta_{\bar{2}} = \bar{\theta}^2 = 0} \quad (\text{B.24})$$

in a *special coordinate system* specified below. Given a gauge-covariant operator of the form $\mathcal{D}_{\mathcal{A}_1} \dots \mathcal{D}_{\mathcal{A}_n}$, its projection $(\mathcal{D}_{\mathcal{A}_1} \dots \mathcal{D}_{\mathcal{A}_n})|$ is defined as follows:

$$\left((\mathcal{D}_{\mathcal{A}_1} \dots \mathcal{D}_{\mathcal{A}_n})|U \right)| := (\mathcal{D}_{\mathcal{A}_1} \dots \mathcal{D}_{\mathcal{A}_n} U)| , \quad (\text{B.25})$$

²⁴In the case of $\mathcal{N} = 2$ AdS Killing vector fields, their $\mathcal{N} = 1$ reduction was carried out in [26].

with U an arbitrary tensor superfield. The conceptual possibility to have a well-defined $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ AdS superspace reduction was noticed in [26]. Specifically, with the choice (B.18) and (B.19), it follows from (B.2) that the operators $(\mathcal{D}_a, \mathcal{D}_\alpha^1, \bar{\mathcal{D}}_{\underline{1}}^{\dot{\alpha}})$ form a closed algebra which is isomorphic to that of the covariant derivatives for $\mathcal{N} = 1$ AdS superspace, eq. (A.2).

As argued in [26], the freedom to perform general coordinate, local Lorentz and $U(1)$ transformations can be used to choose the gauge

$$\mathcal{D}_\alpha^1| = \mathcal{D}_\alpha, \quad \bar{\mathcal{D}}_{\underline{1}}^{\dot{\alpha}}| = \bar{\mathcal{D}}^{\dot{\alpha}}, \quad (\text{B.26})$$

with $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}})$ the covariant derivatives for $\mathcal{N} = 1$ AdS superspace introduced in Appendix A. In such a coordinate system, the operators $\mathcal{D}_\alpha^1|$ and $\bar{\mathcal{D}}_{\dot{\alpha}\underline{1}}|$ do not involve any partial derivatives with respect to $\theta_{\underline{2}}$ and $\bar{\theta}^{\underline{2}}$, and therefore, for any positive integer k , it holds that $(\mathcal{D}_{\hat{\alpha}_1} \cdots \mathcal{D}_{\hat{\alpha}_k} U)| = \mathcal{D}_{\hat{\alpha}_1}| \cdots \mathcal{D}_{\hat{\alpha}_k}| U$, where $\mathcal{D}_{\hat{\alpha}} := (\mathcal{D}_\alpha^1, \bar{\mathcal{D}}_{\underline{1}}^{\dot{\alpha}})$ and U is a tensor superfield. We therefore obtain

$$\mathcal{D}_a| = \mathcal{D}_a. \quad (\text{B.27})$$

Introduce the $\mathcal{N} = 1$ projection of the $\mathcal{N} = 2$ superconformal Killing vector (B.5):

$$\xi| = \xi + \xi_{\underline{2}}^\alpha |\mathcal{D}_\alpha^2| + \bar{\xi}_{\dot{\alpha}}^{\underline{2}} |\bar{\mathcal{D}}_{\underline{2}}^{\dot{\alpha}}| \equiv \xi + \rho^\alpha \mathcal{D}_\alpha^2| + \bar{\rho}_{\dot{\alpha}} \bar{\mathcal{D}}_{\underline{2}}^{\dot{\alpha}}|, \quad (\text{B.28})$$

where we have denoted

$$\xi := \xi^a |\mathcal{D}_a| + \xi_{\underline{1}}^\alpha |\mathcal{D}_\alpha| + \bar{\xi}_{\dot{\alpha}}^{\underline{1}} |\bar{\mathcal{D}}^{\dot{\alpha}}| \equiv \xi^a \mathcal{D}_a + \xi^\alpha \mathcal{D}_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}. \quad (\text{B.29})$$

From (B.21) we obtain

$$\rho_\alpha = \mathcal{D}_\alpha \rho, \quad \rho := -\frac{\mu}{2|\mu|^2} \lambda^{\underline{1}\underline{1}}|. \quad (\text{B.30})$$

In accordance with (B.20b) and (B.22), the parameter ρ obeys the constraints

$$(\mathcal{D}^2 - 4\bar{\mu})\rho = 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_\alpha \rho = 0. \quad (\text{B.31})$$

One can show that these equations also imply²⁵

$$(\bar{\mathcal{D}}^2 - 4\mu)\rho = 0, \quad \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \rho = 0, \quad \mathcal{D}_{\alpha\dot{\alpha}} \rho = 0. \quad (\text{B.32})$$

This implies that ρ can be decomposed into real and imaginary parts which each obey (B.31) and (B.32).

²⁵The proof that $\mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \rho = 0$ follows from analyzing the condition $\bar{\mathcal{D}}_{\dot{\alpha}} (\mathcal{D}^2 - 4\mu)\rho = 0$. One finds that $\bar{\mathcal{D}}_{\dot{\alpha}} \rho = i \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^\alpha \rho / 2\bar{\mu}$ from which the new constraint follows. Similarly, one can show $(\bar{\mathcal{D}}^2 - 4\mu)\rho = 0$ by analyzing $\bar{\mathcal{D}}^2 (\mathcal{D}^2 - 4\mu)\rho = 0$.

It turns out that ξ in (B.28) and (B.29) is an $\mathcal{N} = 1$ superconformal Killing vector field. The simplest way to see that is to rewrite eq. (B.4) in the form

$$0 = [\xi + \frac{1}{2}\lambda^{\text{cd}}M_{\text{cd}}, \mathcal{D}_\alpha^1] + (\frac{1}{2}\bar{\sigma} - \lambda^{\underline{12}})\mathcal{D}_\alpha^1 + (\mathcal{D}^{\beta 1}\sigma)M_{\alpha\beta} \\ + \lambda^{\underline{11}}\mathcal{D}_\alpha^2 + (\mathcal{D}_\alpha^2\sigma - \mathcal{D}_\alpha^1\lambda^{\underline{22}})J_{\underline{22}} - \mathcal{D}_\alpha^1\lambda^{\underline{11}}J_{\underline{11}} , \quad (\text{B.33})$$

where we have used eq. (B.20c). Upon $\mathcal{N} = 1$ projection, this master equation can be seen to lead to completely decoupled conditions on the $\mathcal{N} = 1$ parameters ξ and ρ^α in (B.28) and (B.29). The vector field ξ obeys the $\mathcal{N} = 1$ superconformal Killing equation (A.5) if we identify

$$\lambda^{\text{ab}} := \lambda^{\text{ab}}| , \quad \sigma := \sigma| , \quad \lambda^{\underline{12}} = \frac{1}{2}(\sigma - \bar{\sigma}) . \quad (\text{B.34})$$

The second and third relations hold in the absence of a shadow chiral rotation.

Now we are in a position to list transformations of matter superfields in $\mathcal{N} = 1$ AdS superspace which are generated by the $\mathcal{N} = 2$ superconformal Killing vector ξ . They are:

1. An $\mathcal{N} = 1$ superconformal transformation. It corresponds to the choice

$$\lambda^{\underline{11}}| = 0 , \quad \lambda^{\underline{12}}| = \frac{1}{2}(\sigma - \bar{\sigma})| . \quad (\text{B.35})$$

One can see that all terms in the second line of (B.33) vanish.

2. An extended superconformal transformation. It corresponds to the choice

$$\xi = 0 , \quad \sigma| = 0 , \quad \lambda^{\underline{12}}| = 0 . \quad (\text{B.36})$$

It is described by the complex parameter ρ which is defined by (B.30) and obeys the constraints (B.31).

3. A shadow chiral rotation. It corresponds to the choice

$$\xi = 0 , \quad \lambda^{\underline{11}}| = 0 , \quad \lambda^{\underline{12}}| = \frac{1}{2}\bar{\sigma}| = -\frac{1}{2}\sigma| = \text{const} . \quad (\text{B.37})$$

This transformation does not act on the coordinates of $\mathcal{N} = 1$ AdS superspace.

It is possible to give an explicit solution for the superconformal $\mathcal{N} = 2$ Killing vector in terms of the three independent $\mathcal{N} = 1$ superfields making up the $\mathcal{N} = 2$ superfield $\xi_{\alpha\dot{\alpha}}$. In addition to the covariant derivatives, we must identify the $\mathcal{N} = 1$

reduction of the other $\mathcal{N} = 2$ superconformal generators. The dilatation operator, which we denote \mathbb{D} , obeys

$$[\mathbb{D}, \mathcal{D}_\alpha^i] = \frac{1}{2} \mathcal{D}_\alpha^i, \quad [\mathbb{D}, \bar{\mathcal{D}}_i^{\dot{\alpha}}] = \frac{1}{2} \bar{\mathcal{D}}_i^{\dot{\alpha}}, \quad (\text{B.38})$$

for $\mathcal{N} = 2$ covariant derivatives and

$$[\mathbb{D}, \mathcal{D}_\alpha] = \frac{1}{2} \mathcal{D}_\alpha, \quad [\mathbb{D}, \bar{\mathcal{D}}^{\dot{\alpha}}] = \frac{1}{2} \bar{\mathcal{D}}^{\dot{\alpha}}, \quad (\text{B.39})$$

for $\mathcal{N} = 1$ covariant derivatives. Clearly these definitions coincide. The Lorentz generator M_{ab} should also work the same way in $\mathcal{N} = 1$ and $\mathcal{N} = 2$. However, the $\mathcal{N} = 1$ $\text{U}(1)_R$ generator \mathbb{J} is identified with a certain linear combination of the $\text{U}(1)_R$ and diagonal part of $\text{SU}(2)_R$ from the $\mathcal{N} = 2$ superconformal algebra,

$$\mathbb{J} = \frac{1}{3} \mathbf{J} + \frac{4}{3} J_{12}. \quad (\text{B.40})$$

Here we recall that \mathbf{J} obeys

$$[\mathbf{J}, \mathcal{D}_\alpha^i] = -\mathcal{D}_\alpha^i, \quad [\mathbf{J}, \bar{\mathcal{D}}_i^{\dot{\alpha}}] = +\bar{\mathcal{D}}_i^{\dot{\alpha}} \quad (\text{B.41})$$

while \mathbb{J} must be chosen to obey

$$[\mathbb{J}, \mathcal{D}_\alpha] = -\mathcal{D}_\alpha, \quad [\mathbb{J}, \bar{\mathcal{D}}_{\dot{\alpha}}] = +\bar{\mathcal{D}}_{\dot{\alpha}}. \quad (\text{B.42})$$

The specific relation given in (B.40) is selected out by the superconformal algebra. Another linear combination yields the so-called shadow chiral rotation

$$\mathbb{S} = \frac{1}{2} \mathbf{J} - J_{12} \quad (\text{B.43})$$

which rotates the second supersymmetry generator while leaving the first fixed,

$$[\mathbb{S}, \mathcal{D}_\alpha^1] = 0, \quad [\mathbb{S}, \mathcal{D}_\alpha^2] = -\mathcal{D}_\alpha^2, \quad (\text{B.44})$$

$$[\mathbb{S}, \bar{\mathcal{D}}_1^{\dot{\alpha}}] = 0, \quad [\mathbb{S}, \bar{\mathcal{D}}_2^{\dot{\alpha}}] = +\bar{\mathcal{D}}_2^{\dot{\alpha}}. \quad (\text{B.45})$$

If a tensor superfield \mathbf{U} is conformally primary, then under an $\mathcal{N} = 2$ superconformal Killing isometry \mathbf{U} transforms as in (B.6). Taking the $\mathcal{N} = 1$ projection of this gives

$$\begin{aligned} \delta U &= - \left(\xi \mathbf{U} + \frac{1}{2} \lambda^{\text{cd}} M_{\text{cd}} \mathbf{U} + \lambda^{jk} J_{jk} \mathbf{U} + \frac{1}{2} (\sigma + \bar{\sigma}) \mathbb{D} \mathbf{U} + \frac{1}{4} (\sigma - \bar{\sigma}) \mathbf{J} \mathbf{U} \right) \Big| \\ &= - \left(\xi U + \frac{1}{2} \lambda^{\text{cd}} M_{\text{cd}} U + \frac{1}{2} (\sigma + \bar{\sigma}) \mathbb{D} U + \frac{3}{4} (\sigma - \bar{\sigma}) \mathbb{J} U \right) \\ &\quad - \rho^\alpha (\mathcal{D}_\alpha^2 U) - \bar{\rho}_{\dot{\alpha}} (\bar{\mathcal{D}}_2^{\dot{\alpha}} U) - \lambda^{11} J_{11} U - \lambda^{22} J_{22} U \\ &\quad - i \alpha \mathbb{S} U. \end{aligned} \quad (\text{B.46})$$

Identifying these two equalities implies that a general $\mathcal{N} = 2$ superconformal Killing isometry decomposes into three independent transformations:

1. An $\mathcal{N} = 1$ superconformal Killing isometry with δU given by (A.6), ξ given in (B.29), and the remaining parameters obeying

$$\lambda_{ab} = \lambda_{ab}| = \mathcal{D}_{[a}\xi_{b]} \quad (B.47)$$

$$\sigma = \frac{2}{3}\sigma| + \frac{1}{3}\bar{\sigma}| + \frac{2}{3}\lambda^{12}| = -\frac{i}{24}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot{\alpha}}\xi^{\dot{\alpha}\alpha} - \frac{i}{12}\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha\xi^{\dot{\alpha}\alpha} \quad (B.48)$$

2. An extended superconformal transformation with

$$\delta U = -\rho^\alpha(\mathcal{D}_\alpha^2 U)| - \bar{\rho}_{\dot{\alpha}}(\bar{\mathcal{D}}_{\dot{\alpha}}^2 U)| - \lambda^{11}\mathbb{J}_{11}U - \lambda^{22}\mathbb{J}_{22}U, \quad (B.49)$$

$$\rho_\alpha = \frac{i}{8}(\bar{\mathcal{D}}_{\dot{2}}^{\dot{\alpha}}\xi_{\alpha\dot{\alpha}})|, \quad \lambda^{11}| = -\frac{i}{16}\mathcal{D}_\alpha(\bar{\mathcal{D}}_{\dot{\alpha}2}\xi^{\dot{\alpha}\alpha})| = -\frac{1}{2}\mathcal{D}^\alpha\rho_\alpha \quad (B.50)$$

As discussed in eq. (B.30), it is possible to define the $\mathcal{N} = 1$ superfield ρ in AdS which simplifies these two formulae. From an $\mathcal{N} = 1$ perspective, ρ is an independent superfield.

3. A shadow chiral rotation with

$$\delta U = -i\alpha\mathbb{S}U, \quad (B.51)$$

$$\alpha = -\frac{i}{3}(\sigma - \bar{\sigma})| + \frac{2i}{3}\lambda^{12}| = \frac{1}{96}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]\xi^{\dot{\alpha}\alpha} + \frac{1}{32}([\mathcal{D}_\alpha^2, \bar{\mathcal{D}}_{\dot{\alpha}2}]\xi^{\dot{\alpha}\alpha})|. \quad (B.52)$$

One can check that α is a *constant*. From an $\mathcal{N} = 1$ perspective, it is an independent parameter.

B.3 $\mathcal{N} = 2$ Killing vector fields and their $\mathcal{N} = 1$ reduction

Any superconformal Killing vector field ξ with the additional property

$$\sigma = 0 \quad (B.53)$$

is called a Killing vector field of the $\mathcal{N} = 2$ AdS superspace. The Killing vector obeys the equation

$$[\xi + \frac{1}{2}\lambda^{cd}M_{cd} + \lambda^{kl}J_{kl}, \mathcal{D}_\alpha^i] = 0 \implies \lambda^{kl} = 2\varepsilon S^{kl}, \quad \bar{\varepsilon} = \varepsilon. \quad (B.54)$$

Making use of eq. (B.11) gives

$$\varepsilon = \frac{1}{8}S^{ij}\mathcal{D}_{\alpha i}\xi_j^\alpha. \quad (B.55)$$

As before, we fix S^{ij} as in eqs. (B.18) and (B.19), and thus

$$\lambda^{11} = -2\bar{\mu}\varepsilon. \quad (B.56)$$

The Killing vector fields prove to generate the isometry group of the $\mathcal{N} = 2$ AdS superspace, $\text{OSp}(2|4)$. They were studied in detail in [26].

Consider the $\mathcal{N} = 1$ projection of the $\mathcal{N} = 2$ Killing vector

$$\boldsymbol{\xi}| = \xi + \boldsymbol{\xi}_{\underline{2}}^{\alpha}|\mathcal{D}_{\alpha}^2| + \bar{\boldsymbol{\xi}}_{\dot{\alpha}}^2|\bar{\mathcal{D}}_{\underline{2}}^{\dot{\alpha}}| \equiv \xi + \varepsilon^{\alpha}\mathcal{D}_{\alpha}^2| + \bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}_{\underline{2}}^{\dot{\alpha}}| , \quad (\text{B.57})$$

$$\xi := \boldsymbol{\xi}^{\text{a}}|\mathcal{D}_{\text{a}} + \boldsymbol{\xi}_{\underline{1}}^{\alpha}|\mathcal{D}_{\alpha} + \bar{\boldsymbol{\xi}}_{\dot{\alpha}}^1|\bar{\mathcal{D}}^{\dot{\alpha}}| \equiv \xi^{\text{a}}\mathcal{D}_{\text{a}} + \xi^{\alpha}\mathcal{D}_{\alpha} + \bar{\xi}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} . \quad (\text{B.58})$$

One may see that ξ is a Killing vector of the $\mathcal{N} = 1$ AdS superspace, see Appendix A. In accordance with the relations (B.30) and (B.31), we now have

$$\varepsilon_{\alpha} = \mathcal{D}_{\alpha}\varepsilon , \quad \varepsilon := \boldsymbol{\varepsilon}| = \bar{\varepsilon} , \quad (\text{B.59})$$

where the real parameter ε obeys the constraints

$$(\bar{\mathcal{D}}^2 - 4\mu)\varepsilon = (\mathcal{D}^2 - 4\bar{\mu})\varepsilon = 0 , \quad \mathcal{D}_{\alpha}\bar{\mathcal{D}}_{\dot{\alpha}}\varepsilon = \bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_{\alpha}\varepsilon = 0 . \quad (\text{B.60})$$

We see that there are transformations of matter superfields in $\mathcal{N} = 1$ AdS superspace which are generated by the $\mathcal{N} = 2$ superconformal Killing vector $\boldsymbol{\xi}$. They are:

1. An $\mathcal{N} = 1$ AdS transformation. It is described by an $\mathcal{N} = 1$ AdS Killing vector ξ and corresponds to the choice

$$\boldsymbol{\varepsilon}| = 0 . \quad (\text{B.61})$$

2. An extended supersymmetry transformation. It corresponds to the choice

$$\xi = 0 . \quad (\text{B.62})$$

It is described by a real parameter ε subject to the constraints (B.60). This parameter was introduced in [41] where the $\mathcal{N} = 2$ off-shell massless supermultiplet of arbitrary spin in AdS_4 were constructed.

C Direct proof of invariance for general nonlinear σ -models

We presented in subsection 6.4 an elegant proof of invariance which was somewhat indirect and relied upon the existence of quantities such as the superfield B in (6.38).

We present in this appendix a more direct (but rather technical) proof of the same result, namely that the action

$$S = \int d^4x d^4\theta E \mathcal{K}(\phi, \bar{\phi}) \quad (\text{C.1})$$

is invariant under the $\mathcal{N} = 2$ AdS transformations (6.10) provided the conditions (6.29) and (6.30) hold.

We begin by recasting $\delta_\varepsilon S$ in the form

$$\delta_\varepsilon S = \int d^4x d^4\theta E \left(\frac{1}{2} \mathcal{K}_a \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Omega}^a + \frac{1}{2} \varepsilon \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{K}_a \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Omega}^a + \text{c.c.} \right). \quad (\text{C.2})$$

Making use of the conditions (6.29), the second term on the right can be seen to vanish, and thus

$$\delta_\varepsilon S = -\frac{1}{2} \int d^4x d^4\theta E \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}} + \text{c.c.}, \quad (\text{C.3})$$

where $\bar{\Omega}_{\bar{b}} := g_{a\bar{b}} \bar{\Omega}^a$, and $\bar{\Omega}^a$ is given by eq. (6.31). Rewriting this as a chiral integral yields

$$\delta_\varepsilon S = \int d^4x d^2\theta \mathcal{E} \left\{ \frac{1}{8} \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^2 (\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}}) - \frac{3\mu}{4} \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}} - \frac{\mu}{2} \varepsilon \bar{\mathcal{D}}_{\dot{\alpha}} (\bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}}) \right\} + \text{c.c.} \quad (\text{C.4})$$

To evaluate this further, it helps a great deal to make use of reparametrization-covariant derivatives, which ensure that derivatives of superfields are packaged in a convenient manner. (In particular, all factors of the connection Γ^a_{bc} are hidden from view.) We make use of the formalism developed in detail in [77] at the superfield level.²⁶ On any superfield \mathbf{U}^a which transforms as a target-space vector under holomorphic reparametrizations and as an arbitrary tensor under Lorentz transformations, we may define the derivative ∇_A by

$$\nabla_A \mathbf{U}^a := \mathcal{D}_A \mathbf{U}^a + \Gamma^a_{bc} \mathcal{D}_A \phi^b \mathbf{U}^c, \quad \nabla_A \mathbf{U}^{\bar{a}} := \mathcal{D}_A \mathbf{U}^{\bar{a}} + \Gamma^{\bar{a}}_{\bar{b}\bar{c}} \mathcal{D}_A \bar{\phi}^{\bar{b}} \mathbf{U}^{\bar{c}} \quad (\text{C.5})$$

which is reparametrization-covariant. Similarly on a superfield \mathbf{U}_a , one has

$$\nabla_A \mathbf{U}_a := \mathcal{D}_A \mathbf{U}_a - \Gamma^c_{ab} \mathcal{D}_A \phi^b \mathbf{U}_c, \quad \nabla_A \mathbf{U}_{\bar{a}} := \mathcal{D}_A \mathbf{U}_{\bar{a}} - \Gamma^{\bar{c}}_{\bar{a}\bar{b}} \mathcal{D}_A \bar{\phi}^{\bar{b}} \mathbf{U}_{\bar{c}}. \quad (\text{C.6})$$

This is essentially the pullback to superspace of the target-space covariant derivative. (The generalization to a superfield with multiple target-space indices is straightforward.) In particular, the metric $g_{a\bar{b}}$ is covariantly constant under this derivative.

²⁶The formulation is essentially a generalization of that employed at the component level in the textbook [78].

Making use of these new derivatives, the variation of the action can be written

$$\begin{aligned} \delta_\varepsilon S = \int d^4x d^2\theta \mathcal{E} \left\{ -\frac{1}{2}\varepsilon\mu\bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}}) - \mu\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}} \right. \\ \left. + \frac{1}{8}\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\nabla}_{\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}}\bar{\phi}^{\bar{b}}(\nabla_{\bar{b}}\bar{\Omega}_{\bar{a}} - \nabla_{\bar{a}}\bar{\Omega}_{\bar{b}}) + \frac{1}{8}\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\mathcal{D}}_{\dot{\beta}}\bar{\phi}^{\bar{b}}\bar{\mathcal{D}}^{\dot{\beta}}\bar{\phi}^{\bar{c}}\nabla_{\bar{c}}\nabla_{\bar{b}}\bar{\Omega}_{\bar{a}} \right\} + \text{c.c.} \end{aligned} \quad (\text{C.7})$$

At this point, several simplifications occur. We first recall that

$$\nabla_{\bar{b}}\bar{\Omega}_{\bar{a}} = -\omega_{\bar{b}\bar{a}} \quad (\text{C.8})$$

is both antisymmetric and covariantly constant. Several terms then simplify to yield

$$\begin{aligned} \delta_\varepsilon S = \int d^4x d^2\theta \mathcal{E} \left\{ -\frac{\mu}{2}\varepsilon\bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}}) - \mu\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}} \right. \\ \left. - \frac{1}{4}\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\nabla}_{\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}}\bar{\phi}^{\bar{b}}\omega_{\bar{b}\bar{a}} \right\} + \text{c.c.} \end{aligned} \quad (\text{C.9})$$

Now we must go to components using the $\mathcal{N} = 1$ AdS reduction rule (see [79] or standard texts on $\mathcal{N} = 1$ supergravity [38, 78])

$$\int d^4x d^2\theta \mathcal{E} \mathcal{L}_{\text{chiral}} = -\frac{1}{4} \int d^4x e (\mathcal{D}^2 - 12\bar{\mu}) \mathcal{L}_{\text{chiral}} , \quad (\text{C.10})$$

for any covariantly chiral Lagrangian $\mathcal{L}_{\text{chiral}}$. However, before doing so, there are certain steps we may take which will drastically simplify the resulting calculation. First we rewrite $\delta_\varepsilon S$ as²⁷

$$\begin{aligned} \delta_\varepsilon S = \int d^4x d^2\theta \mathcal{E} \left\{ -\frac{\mu}{2}\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}} - \frac{1}{6}\bar{\mathcal{D}}_{\dot{\beta}}\left(\bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\mathcal{D}}^{\dot{\beta}}\bar{\phi}^{\bar{b}}\omega_{\bar{b}\bar{a}}\right) \right. \\ \left. - \frac{\mu}{2}\bar{\mathcal{D}}_{\dot{\alpha}}\left(\varepsilon\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}}\right) \right\} + \text{c.c.} \end{aligned} \quad (\text{C.11})$$

The first term gives simply

$$\int d^4x e \bar{\varepsilon}_{\dot{\alpha}} \left(-\mu\bar{\mu}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\bar{\Omega}_{\bar{a}} + \frac{i}{2}\mu\mathcal{D}^{\dot{\alpha}\alpha}\bar{\phi}^{\bar{a}}\nabla_{\alpha}\bar{\Omega}_{\bar{a}} + \frac{\mu}{8}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\phi}^{\bar{a}}\nabla^2\bar{\Omega}_{\bar{a}} \right) . \quad (\text{C.12})$$

To evaluate the second and third terms of (C.11) requires the AdS identity [79]

$$\int d^4x e (\mathcal{D}^2 - 12\bar{\mu})\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}} = \int d^4x e \bar{\mathcal{D}}_{\dot{\alpha}}(\mathcal{D}^2 - 8\bar{\mu})\bar{\Psi}^{\dot{\alpha}} , \quad (\text{C.13})$$

²⁷Note that the second and third terms involve the usual covariant derivative since their arguments are scalars under reparametrizations.

where we have discarded a total derivative in the final equality. The second term in (C.11) then evaluates to

$$\begin{aligned}
& \frac{1}{3} \int d^4x e \bar{\mathcal{D}}_{\dot{\alpha}} \left(\bar{\varepsilon}_{\dot{\beta}} \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha}^{\dot{\beta}} \bar{\phi}^{\bar{b}} \omega_{\bar{b}\bar{a}} \right) \\
&= -\frac{1}{3} \int d^4x e \left\{ \varepsilon_{\dot{\beta}} \mathcal{D}^{\dot{\alpha}\alpha} (\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha}^{\dot{\beta}} \bar{\phi}^{\bar{b}} \omega_{\bar{b}\bar{a}}) + \varepsilon_{\dot{\beta}} \mathcal{D}^{\dot{\beta}\alpha} (\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha}^{\dot{\alpha}} \bar{\phi}^{\bar{b}} \omega_{\bar{b}\bar{a}}) \right\} \\
&= i \mu \int d^4x e \varepsilon^{\alpha} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha\dot{\alpha}} \bar{\phi}^{\bar{b}} \omega_{\bar{b}\bar{a}}
\end{aligned} \tag{C.14}$$

after integrating by parts. The third term in (C.11) is a bit more complicated. It evaluates to

$$\frac{\mu}{8} \int d^4x e (\mathcal{D}^2 - 12\bar{\mu}) \bar{\mathcal{D}}_{\dot{\alpha}} (\varepsilon \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}}) = \int d^4x e (\bar{\varepsilon}_{\dot{\alpha}} \Psi_1^{\dot{\alpha}} + \varepsilon^{\alpha} \Psi_{2\alpha} + \varepsilon \Psi_3) ,$$

where

$$\begin{aligned}
\Psi_1^{\dot{\alpha}} &:= \frac{\mu}{8} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla^2 \bar{\Omega}_{\bar{a}} + \frac{i}{2} \mu \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} , \\
\Psi_{2\alpha} &:= -\frac{i}{2} \mu \nabla_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\Omega}_{\bar{a}} - \frac{i}{2} \mu \mathcal{D}_{\alpha\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\nabla}^{\dot{\alpha}} \bar{\Omega}_{\bar{a}} - \frac{\mu}{4} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\nabla}_{\dot{\alpha}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} + \frac{\mu}{4} \bar{\nabla}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} , \\
\Psi_3 &:= \frac{\mu}{8} \bar{\nabla} \bar{\mathcal{D}} \bar{\phi}^{\bar{a}} \nabla^2 \bar{\Omega}_{\bar{a}} + \frac{\mu}{8} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\nabla}^{\dot{\alpha}} \nabla^2 \bar{\Omega}_{\bar{a}} + \frac{i}{2} \mu \nabla^{\dot{\alpha}\alpha} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} + \frac{i}{2} \mu \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \bar{\nabla}_{\dot{\alpha}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} .
\end{aligned}$$

Now we put these three terms together. Performing one integration by parts (for the first term in $\Psi_{2\alpha}$) and making use of (C.8) and $\mu \nabla_b \bar{\Omega}_{\bar{a}} = -\bar{\mu} \nabla_{\bar{a}} \Omega_b$, we find

$$\begin{aligned}
\delta_{\varepsilon} S &= \int d^4x e (\bar{\varepsilon}_{\dot{\alpha}} \Psi_1^{\dot{\alpha}} + \varepsilon^{\alpha} \Psi'_{2\alpha} + \varepsilon \Psi'_3) + \text{c.c.} \\
&= \int d^4x e \left\{ \bar{\varepsilon}_{\dot{\alpha}} (\Psi_1^{\dot{\alpha}} + \bar{\Psi}_2^{\dot{\alpha}}) + \varepsilon^{\alpha} (\Psi'_{2\alpha} + \bar{\Psi}'_{1\alpha}) + \varepsilon (\Psi'_3 + \bar{\Psi}'_3) \right\} ,
\end{aligned} \tag{C.15}$$

where

$$\begin{aligned}
\Psi_1^{\dot{\alpha}} &:= \frac{\mu}{4} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla^{\alpha} \mathcal{D}_{\alpha} \phi^b \nabla_b \bar{\Omega}_{\bar{a}} + \frac{\mu}{4} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}^{\alpha} \phi^b \mathcal{D}_{\alpha} \phi^c \nabla_c \nabla_b \bar{\Omega}_{\bar{a}} + i \mu \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha} \phi^b \nabla_b \bar{\Omega}_{\bar{a}} , \\
\Psi'_{2\alpha} &:= -\frac{\bar{\mu}}{4} \bar{\nabla}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha} \phi^b \nabla_{\bar{a}} \Omega_b - \frac{\bar{\mu}}{4} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{b}} \mathcal{D}_{\alpha} \phi^c \nabla_{\bar{b}} \nabla_{\bar{a}} \Omega_c - i \bar{\mu} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{a}} \mathcal{D}_{\alpha\dot{\alpha}} \phi^b \nabla_{\bar{a}} \Omega_b , \\
\Psi'_3 &:= \frac{\mu}{8} \bar{\nabla} \bar{\mathcal{D}} \bar{\phi}^{\bar{a}} \nabla^2 \bar{\Omega}_{\bar{a}} + \frac{\mu}{8} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \bar{\nabla}^{\dot{\alpha}} \nabla^2 \bar{\Omega}_{\bar{a}} + \frac{i}{2} \mu \nabla^{\dot{\alpha}\alpha} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} + \frac{i}{2} \mu \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \bar{\nabla}_{\dot{\alpha}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} .
\end{aligned}$$

By inspection we can see that the coefficients of ε_{α} and $\bar{\varepsilon}_{\dot{\alpha}}$ will cancel. The remaining terms involving ε may be rearranged into

$$\begin{aligned}
\delta_{\varepsilon} S &= \int d^4x e \varepsilon \left\{ \frac{1}{16} [\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}}] \left(\mu \mathcal{D}^{\alpha} \phi^b \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{b}} \nabla_b \bar{\Omega}_{\bar{b}} + \bar{\mu} \mathcal{D}^{\alpha} \phi^b \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{b}} \nabla_{\bar{b}} \Omega_b \right) \right. \\
&\quad + \frac{i}{4} \mu \nabla^{\dot{\alpha}\alpha} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{a}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} + \frac{i}{4} \mu \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{a}} \bar{\nabla}_{\dot{\alpha}} \nabla_{\alpha} \bar{\Omega}_{\bar{a}} \\
&\quad \left. + \frac{i}{4} \bar{\mu} \nabla^{\dot{\alpha}\alpha} \mathcal{D}_{\alpha} \phi^a \nabla_{\alpha} \Omega_a + \frac{i}{4} \bar{\mu} \mathcal{D}^{\dot{\alpha}\alpha} \phi^a \nabla_{\alpha} \bar{\nabla}_{\dot{\alpha}} \Omega_a \right\} ,
\end{aligned} \tag{C.16}$$

where we have again discarded a total derivative. The first line vanishes using the relation $\mu \nabla_b \bar{\Omega}_{\bar{b}} = -\bar{\mu} \nabla_{\bar{b}} \Omega_b$. The remaining terms can be rearranged using the same relation (and after relabelling some indices) to

$$\begin{aligned} \delta_\varepsilon S = \frac{i}{4} \mu \int d^4x \, e \, \varepsilon \Big\{ & \nabla^{\dot{\alpha}\alpha} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{b}} \mathcal{D}_\alpha \phi^b \nabla_b \bar{\Omega}_{\bar{b}} + \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{b}} \nabla^{\dot{\alpha}\alpha} \mathcal{D}_\alpha \phi^b \nabla_b \bar{\Omega}_{\bar{b}} \\ & + \bar{\mathcal{D}}_{\dot{\alpha}} \phi^{\bar{b}} \mathcal{D}_\alpha \phi^b \mathcal{D}^{\dot{\alpha}\alpha} \bar{\phi}^{\bar{c}} \nabla_{\bar{b}} \nabla_b \bar{\Omega}_{\bar{c}} + \bar{\mathcal{D}}_{\dot{\alpha}} \phi^{\bar{b}} \mathcal{D}_\alpha \phi^b \mathcal{D}^{\dot{\alpha}\alpha} \phi^c \nabla_b \nabla_c \bar{\Omega}_{\bar{b}} \Big\} , \end{aligned} \quad (\text{C.17})$$

where all terms have been rewritten to involve only $\bar{\Omega}$. This can be seen to be a total derivative by using the relations

$$\begin{aligned} \nabla_b \nabla_c \bar{\Omega}_{\bar{b}} &= \nabla_c \nabla_b \bar{\Omega}_{\bar{b}} , \\ \mu \nabla_{\bar{b}} \nabla_b \bar{\Omega}_{\bar{c}} &= -\bar{\mu} \nabla_{\bar{b}} \nabla_{\bar{c}} \Omega_b = -\bar{\mu} \nabla_{\bar{c}} \nabla_{\bar{b}} \Omega_b = \mu \nabla_{\bar{c}} \nabla_b \bar{\Omega}_{\bar{b}} . \end{aligned}$$

Thus the action is invariant and the necessary conditions (6.29) and (6.30) are indeed sufficient.

D Improvement transformations for non-minimal supergravity

Within non-minimal AdS supergravity [56], supercurrent conservation in an AdS background naturally takes the form

$$\bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = -\frac{1}{4} \bar{\mathcal{D}}^2 \zeta_\alpha , \quad \mathcal{D}_{(\beta} \zeta_{\alpha)} = 0 . \quad (\text{D.1})$$

This can be derived by considering the linearized coupling of a matter action to non-minimal supergravity, which involves a real supergravity prepotential $H_{\alpha\dot{\alpha}}$ as well as a complex linear compensator Γ obeying

$$(\bar{\mathcal{D}}^2 - 4\mu)\Gamma = 0 . \quad (\text{D.2})$$

The constraint on Γ can be solved by taking $\Gamma = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$ for an unconstrained spinor superfield $\bar{\psi}^{\dot{\alpha}}$ with a gauge invariance $\delta \bar{\psi}^{\dot{\alpha}} = \bar{\mathcal{D}}_{\dot{\beta}} \bar{\Omega}^{\dot{\beta}\dot{\alpha}}$ for $\bar{\Omega}^{\dot{\beta}\dot{\alpha}} = \bar{\Omega}^{\dot{\alpha}\dot{\beta}}$. The linearized action can generically be written

$$S^{(1)} = \int d^4x \, d^4\theta \, E \left(H^{\dot{\alpha}\alpha} J_{\alpha\dot{\alpha}} + \psi^\alpha \zeta_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \right) . \quad (\text{D.3})$$

The gauge invariance of ψ implies that ζ_α obeys the constraint $\mathcal{D}_{(\beta} \zeta_{\alpha)} = 0$. This constraint is solved by $\zeta_\alpha = \mathcal{D}_\alpha \zeta$ with ζ possessing a gauge invariance $\zeta \rightarrow \zeta + \bar{\Lambda}$ for

an antichiral parameter $\bar{\Lambda}$.²⁸ The equation (D.1) arises when we impose invariance under the supergravity gauge transformations

$$\delta H_{\alpha\dot{\alpha}} = \mathcal{D}_\alpha \bar{L}_{\dot{\alpha}} - \bar{\mathcal{D}}_{\dot{\alpha}} L_\alpha, \quad \delta \psi_\alpha = -\frac{1}{4} \bar{\mathcal{D}}^2 L_\alpha. \quad (\text{D.4})$$

Instead of the transformation for ψ_α given by (D.4), one can consider a more general transformation law

$$\delta \psi_\alpha = -\frac{1}{4} \bar{\mathcal{D}}^2 L_\alpha + \frac{\kappa}{4} \bar{\mathcal{D}}^2 L_\alpha + \frac{\kappa}{4} \bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{D}_\alpha \bar{L}_{\dot{\alpha}} \quad (\text{D.5})$$

for some parameter κ (for simplicity κ is chosen real). This may be understood as simply shifting the original field ψ_α by the term $\frac{\kappa}{4} \bar{\mathcal{D}}^{\dot{\alpha}} H_{\alpha\dot{\alpha}}$. The new supercurrent can be shown to obey the conservation equation

$$\bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = \frac{1}{4} (\kappa - 1) \bar{\mathcal{D}}^2 \zeta_\alpha - \frac{\kappa}{4} \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_\alpha \bar{\zeta}^{\dot{\alpha}}. \quad (\text{D.6})$$

This is an equally valid supercurrent conservation equation for non-minimal supergravity.

In [56], we argued that the supercurrent (10.2) was the most general supercurrent in AdS. As a check, we should show that the supercurrent (D.6) can be rewritten in that form. The approach is simple. Letting $\bar{\zeta}_{\dot{\alpha}} = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\zeta}$, which is always possible in AdS, we note that

$$\begin{aligned} \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_\alpha \bar{\zeta}^{\dot{\alpha}} &= -\frac{1}{2} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha \bar{\zeta} + i[\mathcal{D}^{\dot{\alpha}}, \mathcal{D}_{\alpha\dot{\alpha}}] \bar{\zeta} - \frac{1}{2} \mathcal{D}_\alpha \bar{\mathcal{D}}^2 \bar{\zeta} \\ &= -\frac{1}{2} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha \bar{\zeta} - \frac{1}{2} \mathcal{D}_\alpha (\bar{\mathcal{D}}^2 - 4\mu) \bar{\zeta}. \end{aligned} \quad (\text{D.7})$$

Therefore (D.6) can indeed be written

$$\begin{aligned} \bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} &= \mathcal{D}_\alpha X' - \frac{1}{4} \bar{\mathcal{D}}^2 \zeta'_\alpha \\ X' &= \frac{\kappa}{8} (\bar{\mathcal{D}}^2 - 4\mu) \bar{\zeta}, \quad \zeta'_\alpha = (1 - \kappa) \zeta_\alpha - \frac{\kappa}{2} \mathcal{D}_\alpha \bar{\zeta}. \end{aligned} \quad (\text{D.8})$$

Absorbing X' into ζ'_α (which is always possible in AdS), we can write this instead as

$$\begin{aligned} \bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} &= -\frac{1}{4} \bar{\mathcal{D}}^2 \zeta'_\alpha \\ \zeta'_\alpha &= (1 - \kappa) \zeta_\alpha - \frac{\kappa}{8\mu} \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}}. \end{aligned} \quad (\text{D.9})$$

²⁸In fact, in AdS, one can construct a global solution: $\zeta = \mathcal{D}^\alpha \zeta_\alpha / 4\bar{\mu}$.

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